Homework 1: Closed and Convex sets

Exercise 1: Show that

(a) Given a family $\{C_a \mid a \in \mathcal{A}\}$ of closed sets, their intersection set $\cap_{a\in\mathcal{A}} C_a$ is closed.

(b) Given a finite collection $C_1, \ldots, C_m$ of closed sets, their union $\cup_{i=1}^m C_i$ is closed.

Exercise 2: Give an example showing that the sum $C_1 + C_2$ of two closed sets C_1 and C_2 need not be closed.

Exercise 3: Show that

(a) Given a family $\{C_a \mid a \in \mathcal{A}\}$ of convex sets, their intersection $\cap_{a\in\mathcal{A}} C_a$ is convex.

(b) Given a finite collection $C_1, \ldots, C_m$ of convex sets, their sum $C_1 + \ldots + C_m$ is convex.

Exercise 4: Let C be a convex set. Show that its closure $\mathrm{cl}(C)$ is also convex.

Exercise 5: Let A be an $m \times n$ matrix.

(a) Let $X \subseteq \mathcal{R}^n$ be a convex set. Show that the forward image AX is convex, where

$$AX = \{y \in \mathcal{R}^m \mid y = Ax \text{ for some } x \in X\}.$$

(a) Let $Y \subseteq \mathcal{R}^m$ be a convex set. Show that the inverse image $A^{-1}Y$ is convex, where

$$A^{-1}Y = \{x \in \mathcal{R}^n \mid y = Ax \text{ for some } y \in Y\}.$$

Exercise 6: Show that a “norm-cone” is a cone.

Exercise 7: (a) Let $f : \mathcal{R}^n \to \mathcal{R}$ be a convex function and let $\lambda > 0$ be scalar. Show that function λf is convex.

(b) Let $f_1, f_2 : \mathcal{R}^n \to \mathcal{R}$ be convex functions. Show that their sum $f_1 + f_2$ is a convex function.

(c) Let $f : \mathcal{R}^n \to \mathcal{R}$ be convex and A be an $n \times m$ matrix. Show that their composition $f \circ A$ is a convex function.

(d) Let $f_1, \ldots, f_m : \mathcal{R}^n \to \mathcal{R}$ be convex functions. Show that the pointwise-max function

$$F(x) = \max\{f_1(x), \ldots, f_m(x)\}$$

is convex.

Exercise 8: Consider the function

$$f(x) = -\sum_{i=1}^{n} \ln(1 + x_i).$$

What is the domain of the function? Using the second order conditions, show that the function is convex.