Lecture 13
Transportation Model and Its Variants

October 7, 2009
Definition of the Transportation Model

Given $m$ sources and $n$ destinations, the supply at source $i$ is $a_i$ and the demand at destination $j$ is $b_j$. The cost of shipping one unit of goods from source $i$ to destination $j$ is $c_{ij}$. The goal is to minimize the total transportation cost while satisfying all the supply and demand restrictions.
General Model

- Decision variables: $x_{ij}$ the amount shipped from source $i$ to destination $j$.

- Objective function: minimize $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij}$.

- Constraints
  - supply: for each source node $i$, $\sum_{j=1}^{n} x_{ij} = a_{i}$;
  - demand: for each destination node $j$, $\sum_{i=1}^{m} x_{ij} = b_{j}$.

The general transportation model:

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij}$$

s.t.

$$\sum_{j=1}^{n} x_{ij} = a_{i}, \quad i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_{j}, \quad j = 1, 2, \ldots, n$$

$$x_{ij} \geq 0, \quad i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$$
An Example

MG Auto has three plants in Los Angeles, Detroit, and New Orleans, and two major distribution centers in Denver and Miami. The capacity of the three plants during the next quarter are 1000, 1500, and 1200 cars. The quarterly demand in the two distribution centers are 2300 and 1400 cars. The transportation costs per car on the different routes is given in the following table.

<table>
<thead>
<tr>
<th></th>
<th>Denver(1)</th>
<th>Miami(2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Los Angeles(1)</td>
<td>$80</td>
<td>$215</td>
</tr>
<tr>
<td>Detroit(2)</td>
<td>$100</td>
<td>$108</td>
</tr>
<tr>
<td>New Orleans(3)</td>
<td>$102</td>
<td>$68</td>
</tr>
</tbody>
</table>

**NOTE:** In practice, some of the links might not be available. To deal with this situation, we can assign a very big cost on the unavailable links so that the flow on them are zero in the optimal solution.
Balancing the Model

• In the transportation model, the LP is feasible only if the model is balanced, i.e., the total demand equals to the total supply. In fact, in (1) if we sum up all constraints for demands and all constraints for supplies, we get:

\[
\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} b_j.
\]

• If the model is not balanced, we can add a dummy source or a dummy destination to balance the model.
  – If the total demand is strictly great than the total supply, then we add a dummy source with supply \(\sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i\).
  – If the total demand is strictly less than the total supply, then we add a dummy destination with demand \(\sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j\).

• The unit cost from a dummy source (destination) to any destination (source) nodes are 0.
Dummy Nodes: Examples

In the MG model:

- Suppose that the Detroit plant capacity is 1300 cars (instead of 1500) only. The total supply (3500 cars) is less than the total demand (3700 cars). So add a dummy source with supply $3700 - 3500 = 200$ cars. The unit transportation costs form the dummy source to the two destinations are zero.

- Suppose that the demand at Denver is 1900 cars (instead of 2300) only. The total supply (3700 cars) is larger than the total demand (3300 cars). So add a dummy destination with demand $3700 - 3300 = 400$ cars. The unit transportation costs form the three sources to the dummy destination are zero.
Transshipment Problem

We are given $m$ pure supply nodes with demand $a_i$, $n$ pure demand nodes with demand $b_j$, and $\ell$ transshipment nodes. Suppose the unit transportation cost from supply node $i$ to transshipment node $k$ is $c_{ik}$ and the unit transportation cost for transshipment node $k$ to demand node $j$ is $c_{kj}$. The transshipment problem can be formulated as

$$
\min \sum_{i=1}^{m} \sum_{k=1}^{\ell} c_{ik}x_{ik} + \sum_{k=1}^{\ell} \sum_{j=1}^{n} c_{kj}x_{kj}
$$

s.t.

$$
\sum_{k=1}^{\ell} x_{ik} = a_i, \quad i = 1, 2, \ldots, m
$$
$$
\sum_{i=1}^{m} x_{ik} - \sum_{j=1}^{n} x_{kj} = 0, \quad k = 1, 2, \ldots, \ell
$$
$$
\sum_{k=1}^{\ell} x_{kj} = b_j, \quad j = 1, 2, \ldots, n
$$
$$
x_{ik}, x_{kj} \geq 0, \quad i = 1, 2, \ldots, m; k = 1, \ldots, \ell; j = 1, 2, \ldots, n
$$
Products may be shipped through intermediate or transient nodes before reaching the final destination.

There are three layers of nodes in the model: pure supply nodes, pure demand nodes, and transshipment nodes. For example:
Assignment Problem

• Seeking for assigning the job for the best person

• The general model:
  Given $n$ workers and $n$ jobs, the cost of assigning worker $i$ to job $j$ is given by $c_{ij}$. The goal is to find the assignment with the least cost.

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

s.t. $$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, \ldots, n$$

$$x_{ij} \geq 0, \quad i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$$

(2)

• A special case of the transportation problem with $n = m$ and

$$a_i = b_i = 1, \quad i = 1, \ldots n.$$
Boralis manufacture backpacks for serious hikers. The demand for its product occurs during March to June of each year. Boralis estimates the demand for the four months to be 100, 200, 180, and 300 units, respectively. The company uses part-time labor to manufacture the backpacks and, accordingly, its production capacity varies monthly. It is estimated that Boralis can produce 50, 180, 280, and 270 units in March through June. A current month’s demand may be satisfied in one of three ways:

1. Current month’s production.
2. Surplus production in an earlier month.
3. Surplus production in a later month (backordering).

In the first case, the production cost per backpack is $40. The second case incurs an additional holding cost of $.50 per backpack per month. In the third case, an additional penalty cost of $2.00 per backpack is incurred for each month delay. Boralis wishes to determine the optimal production schedule for the four months to minimize the cost.
Think of the production periods and the demand periods as the sources and destinations respectively. Because of inventory and backordering, that the demand of a period can be satisfied by the production of any periods, but the cost is different.

The cost of satisfying the $j$th period’s demand by the $i$th period’s production is given by

$$c_{ij} = \begin{cases} 
\$40 & i = j \\
\$40 + \$.5 \times (j - i) & i < j \\
\$40 + \$2 \times (i - j) & i > j 
\end{cases}$$

The transportation model and the production-inventory model:

<table>
<thead>
<tr>
<th>Transportation</th>
<th>Production-inventory</th>
</tr>
</thead>
<tbody>
<tr>
<td>Source $i$</td>
<td>Production period $i$</td>
</tr>
<tr>
<td>Destination $j$</td>
<td>Demand period $j$</td>
</tr>
<tr>
<td>Supply at source $i$</td>
<td>Production capacity of period $i$</td>
</tr>
<tr>
<td>Demand at destination $j$</td>
<td>Demand of period $j$</td>
</tr>
<tr>
<td>Unit transportation cost</td>
<td>The cost calculated above</td>
</tr>
</tbody>
</table>
Production-Inventory Control

The optimal solution: