Outline

• Network Models
  • Minimal Spanning Tree Problem
  • Shortest Route Problem - TODAY: modeling aspect
  • Maximal-flow Problem
Shortest Path Problem: Form

Given a road network and a starting node $s$, we want to determine the shortest path to all the other nodes in the network (or to a specified destination node).

This is **Shortest Path Problem**

Note that the graph is directed. The weights on the links are **costs**.

We consider several applications.
Example: Equipment Replacement

RentCar is developing a replacement policy for its car fleet for a 4-year period. At the start of the first year, the RentCar must purchase a car. At the start of each subsequent year, a decision can be made as to keep a car or to replace it. The car has to be in service for at least 1 year and no more than 3 years. The replacement cost is shown in the table below as a function of the period when it is purchased and the years kept in operation.

<table>
<thead>
<tr>
<th>Start of a year</th>
<th>Years in operation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>4000</td>
</tr>
<tr>
<td>2</td>
<td>4300</td>
</tr>
<tr>
<td>3</td>
<td>4800</td>
</tr>
<tr>
<td>4</td>
<td>4900</td>
</tr>
</tbody>
</table>

The problem is to determine the best decision that minimizes the total cost incurred over the period of 4 years.
All possible decisions

<table>
<thead>
<tr>
<th>Decision</th>
<th>Cost</th>
<th>Total Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>By a car in years 1,2,3,4</td>
<td>4000 4300 4800 4900</td>
<td>18000</td>
</tr>
<tr>
<td>By a car in years 1,2,3</td>
<td>4000 4300 7100 x</td>
<td>15400</td>
</tr>
<tr>
<td>By a car in years 1,2,4</td>
<td>4000 6200 x 4900</td>
<td>15100</td>
</tr>
<tr>
<td>By a car in years 1,3,4</td>
<td>5400 x 4800 4900</td>
<td>15100</td>
</tr>
<tr>
<td>By a car in years 1,2</td>
<td>4000 8700 x x</td>
<td>12700</td>
</tr>
<tr>
<td>By a car in years 1,3</td>
<td>5400 x 7100 x</td>
<td>12500</td>
</tr>
<tr>
<td>By a car in years 1,4</td>
<td>9800 x x 4900</td>
<td>14700</td>
</tr>
</tbody>
</table>

The problem of determining the optimal decision can be formulated as shortest path!
The best decision corresponds to the shortest path from node 1 to node 5, which is $1 \rightarrow 3 \rightarrow 5$ with the cost of $5400 + 7100 = 12500$. This path corresponds to the decision of purchasing the car in years 1 and 3.
Most Reliable Route

XY drives daily to work. There are several routes that XY can take. Each link is patrolled by police and for each link there is an estimated probability of not being stopped (see figure below). The reliability of a route is the product of the reliability of the links in the route. XY wants to drive fast (speeding) and maximize the probability of not being stopped by police.
**Transform the problem to minimization form**

Let $\mathcal{P}$ be the set of all paths from node 1 to node 7.
Let $p \in \mathcal{P}$ be a path.
Let $\ell \in p$ denote that link $\ell$ is traversed in a path $p$.
The maximum reliable route is the following problem

$$\max_{p \in \mathcal{P}} \prod_{\ell \in p} \pi_\ell$$

By taking $ln$ transformation of the objective, the problem is equivalent to

$$\max_{p \in \mathcal{P}} \sum_{\ell \in p} \ln(\pi_\ell) \quad (1)$$

The problems are equivalent in the sense that their solutions are the same
The maximization of $\sum_{\ell \in p} \ln(\pi_\ell)$ over $p \in \mathcal{P}$ is equivalent to minimization of $-\sum_{\ell \in p} \ln(\pi_\ell)$ over $p \in \mathcal{P}$, i.e., the problem in (1) is equivalent to

$$\min_{p \in \mathcal{P}} \sum_{\ell \in p} - \ln(\pi_\ell)$$

Again, the equivalence is in the sense that their optimal solutions are the same.

Thus the most reliable route can be found by finding the shortest route in the network, where a link reliability is replaced with $-\ln$ of the reliability (see the network below).
The most reliable route of the original network is the shortest path in this network.
Three-Jug Puzzle

We have an 8-gallon jug filled with water. We also have two empty jugs, one 5-gallon and one 3-gallon. We want to divide 8 gallons of water into two 4-gallon parts using the three jugs. No other measuring devices are available.

What is the smallest number of transfers needed to achieve the result?

We can formulate the problem as shortest path.

- Construct a network with each node representing the amount of water in the jugs
  - a node is an ordered tuple representing the amount of water in 8-gallon, 5-gallon, and 3-gallon jug.
- Place a link from node $a$ to node $b$ when it is possible to move from node $a$ to node $b$ in one transfer (i.e., by pouring water from one jug to another jug)
Each link counts for one transfer. Assign cost of 1 to each link, and find the shortest path from node (8,0,0) to (4,4,0).

The shortest path is shown below. The minimal number of transfers is 7.