Lecture 21
Max-Flow Min-Cut
Integer Linear Programming

October 30, 2009
Outline

- Min-Cut

- Max-Flow Min-Cut Relation

- Integer Linear Programming
Notion of a Cut
We are given a directed capacitated network $G = (V, E, C)$

A cut is defined by a set of nodes $S \subset V$ and it consists of links $(i, j) \in E$ such that $i \in S$ and $j \notin S$.

A cut induced by a node set $S$ is denoted by $C(S)$

$S = \{2, 4\}$

The corresponding cut is

$$C(S) = \{(2, 3), (2, 5), (4, 3), (4, 5)\}$$
\[ S = \{2, 3, 4\} \]

The corresponding cut is

\[ C(S) = \{(3, 5), (4, 5)\}. \]

If the graph was undirected:

\[ C(\{2, 4\}) = \{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{2, 5\}, \{4, 3\}, \{4, 5\}\} \]

\[ C(\{2, 3, 4\}) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{3, 5\}, \{4, 5\}\}. \]
Min-Cut Problem

Given a directed capacitated network $G = (V, E, C)$, find a cut $C(S)$ [set of nodes $S \subset V$] such that the value

$$\text{capacity of the cut } C(S): \sum_{(i,j) \in E \atop i \in S, j \notin S} c_{ij}$$

is the smallest. This is the **min-cut problem**.

- The search for the smallest cut is over all subsets $S \subset V$.

- The number of such subsets can be very large and we cannot search through all of them.

- Efficient algorithm is needed.
s-t Cut

In a given network $G = (V, E, C)$ with designated nodes $s$ and $t$, a cut $S$ such that either $s \in S$ or $t \in S$ is known as s-t cut.

In the preceding figure the cuts $C(\{2, 3\})$ and $C(\{2, 3, 4\})$ are examples of 1-2 cuts.
Max-Flow and Min-Cut Relation

A max-flow in a network with a source node \( s \) and sink node \( t \) is equal to the minimum \( s-t \) cut.

Hence:

- The minimum \( s-t \) cut can be determined by the max-flow algorithm.

- It is one of the \( s-t \) cuts obtained at the last iteration of the max-flow algorithm.
Example
The minimum $s$-$t$ cut is the cut obtained at the end of max-flow algorithm $S = \{s\}$ or $S = \{1, 2, 3, 4, t\}$. The value of the minimum cut is 6.
Application: IAW

IAW is a company that provides fresh water in a given area. The given graph shows the IAW water pipeline system from origin $O$ to a town $T$.

IAW needs to increase its supply of water by 10% due to increase in water demand. The cost of expanding the capacity of a link is proportional to the current link capacity.

What is the best choice of investment for IAW?

In other words what are the pipelines that IAW should invest in and how much per line?
Integer Linear Programming

• Chapter 9

Integer linear programs (ILPs) are linear programs with (some of) the variables being restricted to integer values.

For example

\[
\begin{align*}
\text{max} \quad & 3x_1 + 4x_2 - 6x_3 \\
\text{s.t.} \quad & x_1 + x_2 - x_4 \geq 7 \\
& x_1 + 2x_2 + 4x_3 = 3 \\
& x_1, x_2, x_3 \geq 0 \\
& x_1, x_2, x_3 \text{ are integers}
\end{align*}
\]

pure integer linear program

\[
\begin{align*}
\text{min} \quad & 2x_1 + 9x_2 - 5x_3 \\
\text{s.t.} \quad & 4x_1 + x_2 - 6x_4 = 6 \\
& 2x_1 + x_2 + x_3 \leq 10 \\
& x_1, x_2, x_3 \geq 0 \\
& x_1 \in \{0, 1\}
\end{align*}
\]

mixed integer linear program
Example: Project Selection

Five projects are being evaluated over a 3-year planning horizon. The following table gives the expected returns for each project and the associated yearly expenditures.

<table>
<thead>
<tr>
<th>Project</th>
<th>Expenditures</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>1</td>
<td>8</td>
<td></td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>7</td>
<td>10</td>
<td></td>
<td>40</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>9</td>
<td>2</td>
<td></td>
<td>20</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>4</td>
<td>1</td>
<td></td>
<td>15</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>6</td>
<td>10</td>
<td></td>
<td>30</td>
</tr>
<tr>
<td>Available funds</td>
<td>25</td>
<td>25</td>
<td>25</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Which projects should be selected to maximize the return?
Suppose that due to the nature of the projects we have the following additional instructions

- At most one of projects 1, 3, and 5 can be selected
  - **Mutually exclusive decisions**

- Project 2 cannot be selected unless both projects 3 and 4 are selected
  - **Dependent decisions**
If we relax the ILP to an LP, namely, replace $x_j \in \{0, 1\}$ by $0 \leq x_j \leq 1$ for all $j$, then the optimal solution is $x_1 = 0.5789$, $x_2 = x_3 = x_4 = 1$, $x_5 = 0.7368$. This solution is not suitable for this “yes-no” decision.

If we round the fractional values, we get all variables equal to 1. BUT this is not a feasible solution.

Therefore, the rounding procedure is not a good idea.