Hegselmann-Krause Dynamics: An Upper Bound on Termination Time

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***Abstract*— We discuss the Hegselmann-Krause model for opinion dynamics in discrete-time for a collection of homogeneous agents. Each agent opinion is modeled by a scalar variable, and the neighbors of an agent are defined by a symmetric confidence level. We provide an upper bound for the termination time of the dynamics by using a Lyapunov type approach. Specifically, through the use of a properly defined adjoint dynamics, we construct a Lyapunov comparison function that decreases along the trajectory of the opinion dynamics. Using this function, we develop a novel upper bound on the termination time that has order m^2 in terms of the spread in the initial opinion profile.**

I. Introduction

Recently, mathematical modeling of social networks has gained a lot of attention and several models for opinion dynamics have been proposed and studied (see for example [7], [10], [4]). These models have also been used in distributed engineered systems to capture the dynamic interactions among the autonomous agents such as in robotic networks [3]. One of the most popular models for opinion dynamics is the Hegselmann-Krause proposed by R. Hegselmann and U. Krause in [7] and its gossip-type variant [5], commonly referred to as Deffuant-Weisbuch model.

In this paper we consider the Hegselmann-Krause model for a collection of agents with the same (symmetric) confidence interval. Our interest is in a discrete-time scalar model where the agents' opinions are modeled by scalar variables. The stability and convergence of the Hegselmann-Krause dynamics and its generalizations have been shown in [9], [10], and [11]. Also, several generalizations of this dynamics for continuous time and time-varying networks have been proposed [6], [1], [2], [8].

Among many issues that have remained opened, even for the original Hegselmann-Krause dynamics, is the provision of bounds on the termination time of the dynamics, which is the time required for the dynamics to reach its steady state. While it is known [1] that the termination time is finite, the bounds on this time have not been fully investigated. In [3], it is shown that the termination time of the dynamics is at least in the order of m and at most in the order of m^5, where m is the number of agents in the model. In this work, we further investigate the termination time and provide an upper bound which if of order m^2 in terms of the initial opinion profile of the agents.

The structure of the paper is as follows: in Section II we discuss the Hegselmann-Krause dynamics as appeared in [7], and provide our notation. Then, in Section III, we introduce the concept of *adjoint dynamics* which plays a central role in our analysis. Based on the properties of the dynamics and the use of Lyapunov comparison function, we develop an upper bound for the convergence time in Section IV. We conclude our discussion in Section V.

II. Hegselmann-Krause Opinion Dynamics

Here, we discuss the Hegselmann-Krause opinion dynamics model [7], where m agents interact in time. We use $1, 2, \cdots, m$ to index the agents and let $[m] = \{1, \ldots, m\}$. The interactions occur at time instances indexed by nonnegative integers $t = 0, 1, 2, \ldots$. At each time t, agent i has an opinion that is represented by a scalar $x_i(t) \in \mathbb{R}$. The collection of profiles $\{x_i(t) \mid i \in [m]\}$ is termed the *opinion profile* at time t.

Initially, each agent $i \in [m]$ has an opinion $x_i(0) \in \mathbb{R}$. At each time t, the agents interact with their neighbors and update their opinions, where the neighbors are determined based on the difference of the opinions and a prescribed maximum difference $\epsilon > 0$. The scalar ϵ is termed *confidence*, which limits the agents' interactions in time. Specifically, the set of neighbors of agent i at time t, denoted by $N_i(t)$, is given as follows:

$$N_i(t) = \{j \in [m] \mid |x_i(t) - x_j(t)| \leq \epsilon\}.$$

The opinion profile evolves in time according to the following dynamics:

$$x_i(t+1) = \frac{1}{|N_i(t)|} \sum_{j \in N_i(t)} x_j(t) \qquad \text{for } t \geq 0, \tag{1}$$

where $|N_i(t)|$ denotes the cardinality of the set $N_i(t)$. Note that the opinion dynamics is completely determined by the confidence value ϵ and the initial profile $\{x_i(0) \mid i \in [m]\}$.

First, to compactly represent the dynamics, we define the matrix $A(t)$ with entries given by

$$A_{ij}(t) = \begin{cases} \frac{1}{|N_i(t)|} & \text{if } j \in N_i(t), \\ 0 & \text{otherwise.} \end{cases}$$

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The dynamics (1) can now be written as follows:

$$x(t+1) = A(t)x(t), \tag{2}$$

where $x(t)$ is the column vector with entries $x_i(t)$, $i \in [m]$.

It is well known that Hegselmann-Krause opinion dynamics is stable and converges to a limit point in a finite time [1], [7]; see also [9], [10], [3]. A lower bound and an upper bound for the termination time have been discussed in [3], where it has been demonstrated that a lower bound is at least of the order of the number m of agents, and the upper bound is of the order at most m^5. In this paper, we are interested in improving the upper bound.

Few words on our notation are in place. We view vectors as columns, and use x' to denote the transpose of a vector x and A' for the transpose of a matrix A. We write e_i to denote the unit vector with ith coordinate equal to 1, and e to denote a vector with all entries equal to 1. For a vector x, we use x_i or $[x]_i$ to denote its ith coordinate value. A vector v is stochastic if $v_i \geq 0$ and $\sum_i v_i = 1$. Given a vector $v \in \mathbb{R}^m$, $\mathrm{diag}(v)$ denotes the $m \times m$ diagonal matrix with v_i, $i \in [m]$, on its diagonal. For a matrix A, we write A_{ij} or $[A]_{ij}$ to denote its ijth entry. A matrix A is stochastic if $Ae = e$ and $A_{ij} \geq 0$ for all i, j. We often abbreviate double summation $\sum_{i=1}^m \sum_{j=1}^m$ with $\sum_{i,j=1}^m$. We write $|S|$ to denote the cardinality of a set S with finitely many elements.

III. Adjoint Dynamics

We next discuss the adjoint dynamics for the Hegselmann-Krause dynamics. It has been shown in [12] that, for the Hegselmann-Krause dynamics, there exists a sequence of stochastic vectors $\{\pi(t)\}$ such that

$$\pi'(t) = \pi'(t+1)A(t) \qquad \text{for all } t \geq 0. \tag{3}$$

This dynamics is the *adjoint* for the original dynamics (1). Note that the adjoint dynamics evolves backward in time.

It can be seen that the adjoint dynamics for the Hegselmann-Krause model is not necessarily unique. However, some properties characterize all of possible adjoint dynamics. In particular, one of the properties that is important in the further development is captured by the following relation:

$$\pi'(t+1)x(t+1) = \pi'(t)x(t) \qquad \text{for all } t \geq 0, \tag{4}$$

which is valid for the dynamics $\{x(t)\}$ and any of its adjoint dynamics $\{\pi(t)\}$. Relation (4) is obtained directly from the definitions of the Hegselmann-Krause dynamics in (2) and its adjoint dynamics in (3).

Now, we construct an adjoint dynamics $\{\pi(t)\}$ that is particularly useful for our development of an upper bound on the termination time. We let T be the termination time of the dynamics $\{x(t)\}$, which is the instance when all agents reach their corresponding steady-state opinions. Formally, the termination time T of the dynamics in (2) is the smallest $t \geq 0$ such that $x_i(k+1) = x_i(k)$ for all $k \geq t$ and for all $i \in [m]$, i.e.,

$$T = \min_{t \geq 0} \{t \mid x_i(k+1) = x_i(k) \text{ for all } i \in [m] \text{ and all } k \geq t\}.$$

As shown in [1], the termination time T is finite.

We can define the termination time of the dynamics for any agent $i \in [m]$. For this, at time $k \geq 0$, let the connectivity graph be the graph on the vertex set $[m]$ and the edge set

$$E(k) = \{\{i, j\} \mid i, j \in [m],\ |x_i(k) - x_j(k)| \leq \epsilon\}.$$

For $i \in [m]$, we let $C_i(k)$ be the connected component that contains agent i, i.e. the subset of agents that there is a path between i and any $j \in C_i(k)$. Based on this, we define

$$T_i = \min_{t \geq 0} \{x_i(t) = x_j(t) \text{ for all } j \in C_i(t)\}.$$

In other words T_i is the first time that all the agents in the connected component containing the ith agent reach an agreement.

It can be shown that in fact $T = \max_{i \in [m]} T_i$. Let $S^* \subseteq [m]$ be the index set of the agents whose termination time is the same as T, i.e.,

$$S^* = \{i \in [m] \mid T_i = T\}.$$

Now, we define a special adjoint dynamics. Let $\hat{\pi}(T)$ be given by

$$\begin{aligned} &\hat{\pi}_i(T) = \frac{1}{|S^*|} \qquad \text{for } i \in S^*, \\ &\hat{\pi}_i(T) = 0 \qquad \text{for } i \notin S^*. \end{aligned} \tag{5}$$

Consider the adjoint process $\{\hat{\pi}(t)\}$ defined by

$$\begin{aligned} &\hat{\pi}'(t) = \hat{\pi}'(t+1)A(t) \qquad \text{for } t = 0, \ldots, T-1, \\ &\hat{\pi}(t) = \hat{\pi}(t+1) \qquad \text{for } t \geq T. \end{aligned} \tag{6}$$

Basically, the adjoint dynamics $\{\hat{\pi}(t)\}$ is static after time T, while for times t with $0 \leq t \leq T$, it is constructed backward in time starting with $\hat{\pi}(T)$ as an initial vector. We note that the vectors $\hat{\pi}(t)$ are stochastic since $\hat{\pi}(T)$ is a probability vector and each $A(t)$ is a stochastic matrix.

We have the following result for $\{\hat{\pi}(t)\}$, which will be useful in the further development.

Lemma 1: Consider the adjoint dynamics in (5)–(6). For every agent $i \in [m]$ with $T_i < T$ the following holds:

$$\hat{\pi}_i(t) = 0 \qquad \text{for all } t \geq T_i.$$

Proof: Let i be an agent whose termination time T_i is such that $T_i < T$. Since the dynamics for this agent terminates at time T_i, his neighbor set must have stabilized, i.e., we must have

$$N_i(t) = N_i(T_i) \qquad \text{for all } t \geq T_i.$$

For convenience let us denote the set $N_i(T_i)$ by S. Then, the dynamics of all agents $j \in S$ also terminates at time T_i, and

$$N_j(t) = S \qquad \text{for all } t \geq T_i \text{ and all } j \in S. \tag{7}$$

Since $T_i < T$, it follows that $N_j(T) = S$ for all $j \in S$. Furthermore, we must have $S \cap S^* = \emptyset$ by the definition of the set S^*. Then, by the definition of the vector $\hat{\pi}(T)$ in (5), it follows that

$$\hat{\pi}_j(T) = 0 \qquad \text{for all } j \in S. \tag{8}$$

Now, since the adjoint dynamics $\hat{\pi}(t)$ is static after time T, we immediately have

$$\hat{\pi}_j(t) = 0 \qquad \text{for all } j \in S \text{ and all } t \geq T.$$

We now show that $\hat{\pi}_j(t) = 0$ for all $j \in S$ and for all times t with $T_i \leq t < T$. We prove this by induction on t. For $t = T$, the relation holds as seen from (8). Assume now that, the relation holds for some $t+1$ with $T_i \leq t+1 \leq T$, and consider the preceding time t. By the definition of $\hat{\pi}(t)$ in (6), we have for any $j \in S$

$$\hat{\pi}_j(t) = \sum_{\ell \in N_j(t)} \frac{\hat{\pi}_\ell(t+1)}{|N_\ell(t)|} = \sum_{\ell \in S} \frac{\hat{\pi}_\ell(t+1)}{|S|} = 0,$$

where the second equality follows from (7), while the last equality follows by the induction hypothesis. Thus, we have

$$\hat{\pi}_j(t) = 0 \qquad \text{for all } j \in S \text{ and all } t \geq T_i.$$

The stated result follows since $i \in S$. ■

The following result will also be used in the further development. The result provides a necessary and sufficient condition for an agreement of two agents.

Lemma 2: Let $x(0) \in \mathbb{R}^m$ be ordered so that $x_1(0) \leq x_2(0) \leq \cdots \leq x_m(0)$. Then, we have $x_i(t+1) = x_{i+1}(t+1)$ for some t and $i \leq m-1$ if and only if $N_i(t) = N_{i+1}(t)$.

Proof: The "if part" of the statement follows immediately from the definition of the dynamics in (2). To show the "only if" part, let the average of the opinions $\{x_i(t) \mid p \leq i \leq q\}$, with $p \leq q$, be denoted by $a_p^q(t)$, i.e.,

$$a_p^q(t) = \frac{1}{q-p+1} \sum_{i=p}^{q} x_i(t).$$

Then, for $p \leq q < m$, we have $a_p^q(t) \leq x_{q+1}(t)$ and hence:

$$\begin{aligned} a_p^q(t) &= \left(\frac{q-p+1}{q-p+2} + \frac{1}{q-p+2}\right) a_p^q(t) \\ &\leq \frac{q-p+1}{q-p+2}\, a_p^q(t) + \frac{1}{q-p+2} x_{q+1}(t) \\ &= \frac{q-p+1}{q-p+2} \frac{1}{q-p+1} \sum_{i=p}^{q} x_i(t) + \frac{1}{q-p+2} x_{q+1}(t) \\ &= a_p^{q+1}(t). \end{aligned}$$

Thus $a_p^q(t) \leq a_p^{q+1}(t)$. Further, note that if $x_q(t) < x_{q+1}(t)$, then $a_p^q(t) < a_p^{q+1}(t)$. Similarly, we have $a_{p-1}^q(t) \leq a_p^q(t)$ for $1 < p \leq q \leq m$, and if $x_{p-1}(t) < x_p(t)$, then $a_{p-1}^q(t) < a_p^q(t)$. Putting these relations together, we obtain for $p \leq q$,

$$a_{p-1}^q(t) \leq a_p^q(t) \leq a_p^{q+1}(t), \tag{9}$$

where each of the inequalities is strict if in the profile ordering $x_{p-1}(t) \leq x_p(t) \leq x_{p+1}(t)$ the corresponding inequality is strict.

Now, let $x_i(t+1) = x_{i+1}(t+1)$. Note that in this case we must have $N_i(t) \cap N_{i+1}(t) \neq \emptyset$; for otherwise we would have $x_i(t+1) \leq x_i(t) < x_{i+1}(t) \leq x_{i+1}(t+1)$, thus contradicting $x_i(t+1) = x_{i+1}(t+1)$. Let us assume that $x_i(t+1) = a_{p_1}^{q_1}(t)$ for some $p_1 \leq q_1$ and $x_{i+1}(t+1) = a_{p_2}^{q_2}(t)$ for some $p_2 \leq q_2$. Since $N_i(t) \cap N_{i+1}(t) \neq \emptyset$, we must have $p_2 \leq q_1$, implying $p_1 \leq p_2 \leq q_1 \leq q_2$. Thus, using the inequalities in (9), we obtain

$$x_i(t+1) = a_{p_1}^{q_1}(t) \leq a_{p_2}^{q_1}(t) \leq a_{p_2}^{q_2}(t) = x_{i+1}(t+1).$$

Note that if $N_i(t) \neq N_{i+1}(t)$, then at least one of the above inequalities is strict, because in this case either $x_{p_1}(t) < x_{p_2}(t)$ or $x_{q_1}(t) < x_{q_2}(t)$. Thus, the claim follows. ■

We next establish an important property of the adjoint dynamics $\{\hat{\pi}(t)\}$ for later development. The result makes use of Lemmas 1 and 2. From now on, *we assume that* $T \geq 2$.

Lemma 3: Let $\{\hat{\pi}(t)\}$ be the adjoint dynamics as given in (6), and consider a time $t \geq 0$ such that $t < T-1$. Then, for any $i \in [m]$ with $\hat{\pi}_i(t+1) > 0$, there exists $j \in N_i(t)$ such that $\hat{\pi}_j(t+1) \geq \frac{1}{2}\hat{\pi}_i(t+1)$ and $N_j(t) \neq N_i(t)$.

Proof: Without loss of generality, we assume that the ordering of the initial profile is $x_1(0) \leq x_2(0) \leq \cdots \leq x_m(0)$, for otherwise we can re-label the agents.

Let us define the closest upper and lower neighbors of an agent $i \in [m]$, respectively, as follows

$$\bar{i}(t) = \min\{j \in N_i(t) \mid x_i(t) < x_j(t)\},$$

$$\underline{i}(t) = \max\{\ell \in N_i(t) \mid x_\ell(t) < x_i(t)\}.$$

Note that the closest upper (lower) neighbor of an agent $i \in [m]$ may not exist. Define the following subsets of $N_i(t+1)$:

$$N_i^+(t+1) = \{j \in N_i(t+1) \mid x_i(t+1) \leq x_j(t+1)\},$$

$$N_i^-(t+1) = \{\ell \in N_i(t+1) \mid x_\ell(t+1) \leq x_i(t+1)\}.$$

By the definition of the adjoint dynamics in (6), we have for $t < T-1$,

$$\begin{aligned} \hat{\pi}_i(t+1) &= \sum_{j \in N_i(t+1)} \hat{\pi}_j(t+2) A_{ji}(t+1) \\ &\leq \sum_{j \in N_i^-(t+1)} \hat{\pi}_j(t+2) A_{ji}(t+1) \\ &\quad + \sum_{j \in N_i^+(t+1)} \hat{\pi}_j(t+2) A_{ji}(t+1), \end{aligned}$$

where the inequality comes from $N_i^-(t+1) \cup N_i^+(t+1) \subset N_i(t+1)$ which holds because i is included in $N_i(t+1)$ but not in the other two sets. Thus, we conclude that either

$$\frac{\hat{\pi}_i(t+1)}{2} \leq \sum_{j \in N_i^-(t+1)} \hat{\pi}_j(t+2) A_{ji}(t+1),$$

or

$$\frac{\hat{\pi}_i(t+1)}{2} \leq \sum_{j \in N_i^+(t+1)} \hat{\pi}_j(t+2) A_{ji}(t+1).$$

Without loss of generality assume that

$$\frac{\hat{\pi}_i(t+1)}{2} \leq \sum_{j \in N_i^+(t+1)} \hat{\pi}_j(t+2) A_{ji}(t+1).$$

Now, let i be such that $\hat{\pi}_i(t+1) > 0$ and consider the following two cases:

Case 1: $\bar{i}(t+1)$ *exists.* Then, we must have $i \leq m-1$.

Furthermore, we have $\bar{i}(t+1) \in N_i^+(t+1)$ and $N_i^+(t+1) \subseteq N_{\bar{i}(t+1)}(t+1)$. Therefore,

$$\begin{aligned}\hat{\pi}_{\bar{i}(t+1)}(t+1) &= \sum_{j \in N_{\bar{i}}(t+1)} \hat{\pi}_j(t+2) A_{j\bar{i}}(t+1) \\ &\geq \sum_{j \in N_i^+(t+1)} \hat{\pi}_j(t+2) A_{j\bar{i}}(t+1) \\ &= \sum_{j \in N_i^+(t+1)} \hat{\pi}_j(t+2) A_{ji}(t+1) \\ &\geq \frac{\hat{\pi}_i(t+1)}{2},\end{aligned}$$

where the first inequality follows from the fact that the positive entries in each row of $A(t+1)$ are identical. The last inequality follows from the definition of the adjoint dynamics in (6).

We next show that $\bar{i}(t+1) \in N_i(t)$. To arrive at a contradiction, assume that $\bar{i}(t+1) \notin N_i(t)$. Since $\bar{i}(t) \in N_i(t)$, it follows that $\bar{i}(t) < \bar{i}(t+1)$. Thus, there must be at least one agent j whose value $x_j(t)$ is between the values $x_i(t)$ and $x_{\bar{i}(t+1)}(t)$. Assume that there are $\tau \geq 1$ agents between i and $\bar{i}(t+1)$, i.e., we have

$$x_i(t) \leq x_{i+1}(t) \leq \cdots \leq x_{i+\tau}(t) < x_{\bar{i}(t+1)}(t), \qquad (10)$$

where the last inequality is strict by $\bar{i}(t+1) \notin N_i(t)$.

At time $t+1$, the same order as in (10) is preserved

$$x_i(t+1) \leq x_{i+1}(t+1) \leq \cdots \leq x_{i+\tau}(t+1) \leq x_{\bar{i}(t+1)}(t+1).$$

By the definition of $\bar{i}(t+1)$ (as the closest neighbor with a larger opinion value than the value of agent i), it follows that

$$x_i(t+1) = x_{i+1}(t+1) = \cdots = x_{i+\tau}(t+1) < x_{\bar{i}(t+1)}(t+1).$$

According to Lemma 2, the profiles are equal at time $t+1$ if and only if

$$N_i(t) = N_{i+1}(t) = \cdots = N_{i+\tau}(t).$$

Since $\bar{i}(t+1) \notin N_i(t)$, it follows that (at time t) the agents in the set $\{i, i+1, \ldots, i+\tau\}$ are connected, but all of them are disconnected from agent $\bar{i}(t+1)$. As a consequence, i and $\bar{i}(t+1)$ cannot be neighbors at time $t+1$, which is a contradiction. Hence, we must have $\bar{i}(t+1) \in N_i(t)$.

Case 2: $\bar{i}(k+1)$ does not exist. Then, for any $j \in N_i^+(t+1)$, we have $x_j(t+1) = x_i(t+1)$. Thus, $N_i^+(t+1) \subseteq N_i^-(t+1)$ and hence, $N_i(t+1) = N_i^-(t+1)$. If $\underline{i}(t+1)$ does not exist, then $x_\ell(t+1) = x_i(t+1)$ for all $\ell \in N_i(t)$. Hence, it follows that $t+1$ is the termination time for agent i. Since $t+1 < T$, by Lemma 1 it follows that $\hat{\pi}_i(t+1) = 0$, which contradicts the assumption $\hat{\pi}_i(t+1) = 0$. Therefore, $\underline{i}(t+1)$ must exist. Since $N_i^-(t+1) \subseteq N_{\underline{i}(t+1)}(t+1)$, using the same line of argument as in the preceding case, we can show that

$$\hat{\pi}_{\underline{i}(t+1)} \geq \frac{1}{2}\hat{\pi}_i(t+1), \qquad N_{\underline{i}(t+1)}(t) \neq N_i(t),$$

implying that the assertion holds for $j = \underline{i}(t+1)$. ■

IV. Upper Bound for Termination Time

In this section, we establish an improved upper bound for the termination time of the Hegselmann-Krause dynamics. Our analysis uses a Lyapunov comparison function that is constructed by using the adjoint dynamics in (5)–(6). However, we start by discussing a Lyapunov function defined by a (generic) adjoint dynamics in (3).

The comparison function is $V_\pi(x,t)$ defined by: for all $x \in \mathbb{R}^n$ and $t \geq 0$,

$$V_\pi(x,t) = \sum_{i=1}^m \pi_i(t)(x_i - \pi'(t)x)^2, \qquad (11)$$

where $\{\pi(t)\}$ is the adjoint dynamics in (3). This function has been proposed and investigated in [12]. In particular, the decrease of this comparison function along trajectories of the Hegselmann-Krause dynamics $\{x(t)\}$ and its adjoint dynamics is of particular importance in our analysis. We use the following result, as shown in [12], Theorem 4.3.

Theorem 1: For any $t \geq 0$, we have

$$\begin{aligned}V_\pi(x(t+1), t+1) &= V_\pi(x(t), t) \\ &\quad - \frac{1}{2}\sum_{i,j=1}^m H_{ij}(t)(x_i(t) - x_j(t))^2,\end{aligned}$$

where $H(t) = A'(t)\,\mathrm{diag}(\pi(t+1))A(t)$.

As an immediate consequence of Theorem 1 and the definition of the adjoint dynamics in (3), by summing both sides of the asserted relation for $t = 0, \ldots, \tau$, we have for all τ, $0 \leq \tau \leq T-1$,

$$\begin{aligned}V_\pi(x(\tau+1), \tau+1) &= V_\pi(x(0), 0) \\ - \frac{1}{2}\sum_{t=0}^{\tau}\sum_{i,j=1}^m &H_{ij}(t)(x_i(t) - x_j(t))^2. \qquad (12)\end{aligned}$$

We next derive a lower bound for the summands that appear with the negative sign in relation (12). We do so through the use of two auxiliary relations. These relations are valid for any adjoint dynamics.

The first relation gives us a lower bound in terms of the maximal opinion spread in the neighborhood for every agent. Specifically, for every $\ell \in [m]$ and every $t \geq 0$, let

$$d_\ell(t) = \max_{i \in N_\ell(t)} x_i(t) - \min_{j \in N_\ell(t)} x_j(t).$$

Thus, $d_\ell(t)$ measures the opinion spread that agent ℓ observes at time t. We have the following result.

Lemma 4: For any adjoint dynamics $\{\pi(t)\}$ of (3) we have for all $t \geq 0$,

$$\sum_{i,j=1}^m H_{ij}(t)(x_i(t) - x_j(t))^2 \geq \frac{1}{4m}\sum_{\ell=1}^m \pi_\ell(t+1)\, d_\ell^2(t).$$

Proof: Since $H(t) = A'(t)\,\mathrm{diag}(\pi(t+1))A(t)$, it follows that $H_{ij}(t) = \sum_{\ell=1}^m \pi_\ell(t+1)A_{\ell i}(t)A_{\ell j}(t)$. Therefore, by exchanging the order of summation we obtain

$$\sum_{i,j=1}^m H_{ij}(t) = \sum_{\ell=1}^m \pi_\ell(t+1)\sum_{i,j=1}^m A_{\ell i}(t)A_{\ell j}(t).$$

Using the definition of the matrix $A(t)$, we obtain

$$A_{\ell i}(t)A_{\ell j}(t) = \begin{cases} \frac{1}{|N_\ell(t)|^2} & \text{if } i,j \in N_\ell(t), \\ 0 & \text{otherwise,} \end{cases}$$

thus implying

$$\sum_{i,j=1}^{m} H_{ij}(t)(x_i(t)-x_j(t))^2 \geq \sum_{\ell=1}^{m} \frac{\pi_\ell(t+1)}{|N_\ell(t)|^2} \sum_{i,j\in N_\ell(t)} (x_i(t)-x_j(t))^2. \quad (13)$$

We next show that for all $t \geq 0$,

$$\sum_{i,j\in N_\ell(t)} (x_i(t)-x_j(t))^2 \geq \frac{1}{4}|N_\ell(t)| d_\ell^2(t).$$

If $d_\ell(t) = 0$, then the assertion follows immediately. So, suppose that $d_\ell(t) \neq 0$. Let us define $lb = \text{argmin}\{x_i(t) \mid i \in N_\ell(t)\}$ and $ub = \text{argmax}\{x_j(t) \mid j \in N_\ell(t)\}$. In other words, lb and ub are the agents with the smallest and largest opinion values that agent ℓ observes in his neighborhood. Therefore, $d_\ell(t) = x_{ub}(t) - x_{lb}(t)$ and since $d_\ell(t) \neq 0$, we have $lb \neq ub$. Letting $\sum_{i,j\in N_\ell(t)}$ denote the double summation $\sum_{i\in N_\ell(t)}\sum_{j\in N_\ell(t)}$, we have

$$\begin{aligned}\sum_{i,j\in N_\ell(t)} (x_i(t)-x_j(t))^2 &\geq \sum_{j\in N_\ell(t)} (x_{ub}(t)-x_j(t))^2 \\ &+ \sum_{i\in N_\ell(t)} (x_i(t)-x_{lb}(t))^2 \\ &= \sum_{j\in N_\ell(t)} \left\{(x_{ub}(t)-x_j(t))^2 + (x_j(t)-x_{lb}(t))^2\right\} \\ &\geq \sum_{j\in N_\ell(t)} \frac{1}{4}(x_{ub}(t)-x_{lb}(t))^2 = \frac{1}{4}|N_\ell(t)| d_\ell^2(t). \quad (14)\end{aligned}$$

The last inequality follows from

$$(x_{ub}(t)-x_j(t))^2 + (x_j(t)-x_{lb}(t))^2 \geq \frac{1}{4}(x_{ub}(t)-x_{lb}(t))^2,$$

which holds since the function $s \to (a-s)^2 + (s-b)^2$ attains its minimum at $s = \frac{a+b}{2}$. By combining relations (13) and (14), we obtain

$$\sum_{i,j=1}^{m} H_{ij}(t)(x_i(t)-x_j(t))^2 \geq \frac{1}{4}\sum_{\ell=1}^{m} \frac{\pi_\ell(t+1)}{|N_\ell(t)|} d_\ell^2(t).$$

The desired relation follows by $|N_i(t)| \leq m$ for all t. ∎

Now, we focus a relation that is valid for the special adjoint dynamics of (5)–(6). Specifically, we estimate the sum $\sum_{\ell=1}^{m} \hat{\pi}_\ell(t+1)\, d_\ell^2(t)$.

Lemma 5: For the dynamics of (5)–(6) we have for all $0 \leq t < T-1$,

$$\sum_{\ell=1}^{m} \hat{\pi}_\ell(t+1)\, d_\ell^2(t) \geq \frac{\epsilon^2}{4m}.$$

Proof: Let $\ell^* = \text{argmax}_{\ell\in[m]}\hat{\pi}_\ell(t+1)$. Since, $\hat{\pi}(t+1)$ is a stochastic vector, it follows that $\hat{\pi}_{\ell^*}(t+1) \geq \frac{1}{m}$. Suppose that $d_{\ell^*}(t) > \frac{\epsilon}{2}$. In this case, it immediately follows

$$\sum_{\ell=1}^{m} \hat{\pi}_\ell(t+1)\, d_\ell^2(t) \geq \hat{\pi}_{\ell^*}(t+1)\, d_{\ell^*}^2(t) > \frac{1}{m}\,\frac{\epsilon^2}{4},$$

thus showing the desired relation.

Suppose now that $d_{\ell^*}(t) \leq \frac{\epsilon}{2}$. Since $t < T-1$, by Lemma 3, there exists $j^* \in N_{\ell^*}(t)$ such that

$$\hat{\pi}_{j^*}(t+1) \geq \frac{1}{2}\hat{\pi}_{\ell^*}(t+1) \geq \frac{1}{2m}.$$

Moreover, $N_{j^*}(t) \neq N_{\ell^*}(t)$. Next, we prove that $d_{j^*}(t) \geq \epsilon$. Since $d_{\ell^*}(t) \leq \frac{\epsilon}{2}$ and $j^* \in N_{\ell^*}(t)$, it follows that $|x_{\ell^*}(t) - x_{j^*}(t)| \leq \frac{\epsilon}{2}$. Also, for any $i \in N_{\ell^*}(t)$, we have:

$$|x_i(t) - x_{j^*}(t)| \leq |x_i(t) - x_{\ell^*}(t)| + |x_{\ell^*}(t) - x_{j^*}(t)| \leq \epsilon,$$

implying $i \in N_{j^*}(t)$. Thus, $N_{\ell^*}(t) \subset N_{j^*}(t)$. Let $i \in N_{j^*}(t) \setminus N_{\ell^*}(t)$. Then, $|x_i(t) - x_{\ell^*}(t)| > \epsilon$, and hence, $d_{j^*}(t) > \epsilon$. Therefore, we obtain

$$\sum_{\ell=1}^{m} \hat{\pi}_\ell(t+1)\, d_\ell^2(t) \geq \hat{\pi}_j^*(t+1)\, d_{j^*}^2(t) \geq \frac{1}{2m}\,\epsilon^2.$$

All in all, in any case for $t < T-1$, we have

$$\sum_{\ell=1}^{m} \hat{\pi}_\ell(t+1)\, d_\ell^2(t) \geq \frac{\epsilon^2}{4m}.$$

∎

We are now ready to provide our main result, which yields an upper bound for the termination time T.

Theorem 2: For the Hegselmann-Krause dynamics we have

$$T - 2 \leq \frac{32m^2}{\epsilon^2}\, V_{\hat{\pi}}(x(0), 0).$$

Proof: From Lemma 4 and relation (12), with $t = T-2$, we obtain

$$\begin{aligned}V_{\hat{\pi}}(x(T-1), T-1) &= V_{\hat{\pi}}(x(0), 0) \\ &- \frac{1}{2}\sum_{t=0}^{T-2}\sum_{i,j=1}^{m} H_{ij}(t)(x_i(t)-x_j(t))^2.\end{aligned}$$

By using Lemma 4, we further obtain

$$\begin{aligned}V_{\hat{\pi}}(x(T-1), T-1) &= V_{\hat{\pi}}(x(0), 0) \\ &- \frac{1}{8m}\sum_{t=0}^{T-2}\sum_{\ell=1}^{m} \hat{\pi}_\ell(t+1) d_\ell^2(t).\end{aligned}$$

Finally, we use Lemma 5 to arrive at

$$V_{\hat{\pi}}(x(T-1), T-1) = V_{\hat{\pi}}(x(0), 0) - \frac{\epsilon^2}{32m^2}\sum_{t=0}^{T-2} 1.$$

Therefore,

$$(T-2)\frac{\epsilon^2}{32m^2} \leq V_{\hat{\pi}}(x(0), 0),$$

and the desired relation follows. ∎

The bound of Theorem 2 shows that the convergence time is of the order of m^2, provided that the value $V_{\hat{\pi}}(x(0), 0)$

does not depend on m. One may further analyze the bound to take $V_{\hat{\pi}}(x(0),0)$ into account. We do so by taking the worst case into consideration.

By further bounding $V_{\hat{\pi}}(x(0),0)$, in the worst case, we have

$$\begin{aligned} V_{\hat{\pi}}(x(0),0) &= \sum_{i=1}^{m} \hat{\pi}_i(0)\left(x_i(0) - \hat{\pi}'(0)x(0)\right)^2 \\ &\leq \sum_{i=1}^{m} \hat{\pi}_i(0)\left(x_i(0) - \min_{j\in[m]} x_j(0)\right)^2 \\ &\leq d^2(x(0)) \sum_{i=1}^{m}\sum_{i=1}^{m} \hat{\pi}_i(0), \end{aligned}$$

where $d(x(0)) = \max_{i\in[m]} x_i(0) - \min_{j\in[m]} x_j(0)$. Since $\hat{\pi}(0)$ is a stochastic vector, we obtain

$$V_{\hat{\pi}}(x(0),0) \leq d^2(x(0)) \leq m^2\epsilon^2.$$

Thus, as the worst case bound, we have the following result

$$T - 2 \leq 32m^4.$$

Thus, $32m^4$ is an upper estimate for the termination time of the Hegselmann-Krause opinion dynamics. The bound is of the order m^4, which is by the factor of m better than the previously known bound [3].

V. Discussion

In this paper, we have considered the Hegselmann-Krause model for opinion dynamic. By choosing an appropriate adjoint dynamics and Lyapunov comparison function, we have established an upper bound for the termination time of the dynamics. For a collection of m agents, our bound is of the order m^2 under some assumptions, while it may be of the order of m^4 in worst case. In the worst case, our bound improves the previously known bound by a factor of m. It however, remains to explore the tightness of these bounds.

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