Constrained Consensus for Bargaining in Dynamic Coalitional TU Games

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***Abstract*— We consider a sequence of transferable utility (TU) games where, at each time, the characteristic function is a random vector with realizations restricted to some set of values. We assume that the players in the game interact only with their neighbors, where the neighbors may vary over time. The game differs from other ones in the literature on dynamic, stochastic or interval valued TU games as it combines dynamics of the game with an allocation protocol for the players that dynamically interact with each other. The protocol is an iterative and decentralized algorithm that offers a paradigmatic mathematical description of negotiation and bargaining processes. The main contributions of the paper are the definition of a robust (coalitional) TU game and the development of a distributed bargaining protocol. We prove the convergence with probability 1 of the bargaining protocol to a random allocation that lies in the core of the robust game under some mild conditions on the players' communication graphs.**

I. INTRODUCTION

Coalitional games with transferable utilities (TU) have been introduced by von Neumann and Morgenstern [24]. They have been used to model cooperation in supply chain or inventory management applications [6], [10], network flow applications [2] and in communication networks [20].

In this paper, we consider a sequence of coalitional TU games for a finite set of players. The game is played repeatedly over time, thus generating a sequence of time varying characteristic functions. We refer to such a repeated game as *dynamic coalitional TU game*. In this setting, a player can observe only the allocations of his neighbors, which may change in time.

We consider bargaining protocols assuming that each player i obeys rationality and efficiency by deciding on an allocation vector which satisfies the value constraints of all the coalitions that include player i. This set is termed *bounding set of player* i. At every iteration, a player i observes the allocations of some of his neighbors. This is modeled using a directed graph with the set of players as the vertex set and a time-varying edge set composed of directed links (i, j) whenever player i observes the allocation vector proposed by player j at time t. We refer to this directed graph as players' *neighbor-graph*. Given a player's neighbor-graph, each player i negotiates allocations by adjusting the allocations he received from his neighbors through weight assignments. As the balanced allocation may violate his rationality constraints (it lies outside player i's bounding set), the player selects a new allocation by projecting the balanced allocation on his bounding set. We propose such bargaining protocols for solving the robust TU game. We use some mild assumptions on the connectivity of the players' neighbor-graph and the weights that the players use when balancing their own allocations with the neighbors' allocations. Assuming that the core of the robust game is nonempty, we show that our bargaining protocol converges with probability 1 to a common (random) allocation in the core.

The work in this paper deviate from the stochastic framework provided in [9], [21], [22] in at least three aspects: i) the existence of a neighbor-graph, ii) the presence of multiple iterations in the bargaining process, iii) and the consideration of the robust game. Also, a new element with respect to previous work [8], [11], is that the values of the coalitions are realized exogenously and no relation is assumed between consecutive samples.

Dynamic robust TU games have been considered in [3] and [4] but for a continuous time setting in the former work and for a centralized allocation process in the latter one. Convergence of allocation processes is a main topic also in [5], [13]. The difference with our approach is that in [5], [13], rewards are allocated by a game designer repeatedly in a centralized manner. Convergence of bargaining processes has also been explored under dynamic coalition formation [1] for a different dynamic model, where players decide both on which coalition to address and what payoff to announce.

The work in this paper is also related to the literature on agreement among multiple agents, where an underlying communication graph for the agents and balancing weights have been used with some variations [23], [15] to reach an agreement on common decision variable, as well as in [16], [17], [19], [18] for distributed multi-agent optimization.

This paper is organized as follows. In Section II, we introduce the dynamic TU game, the robust game and the bargaining protocol for this game. We then give some preliminary results. In Section III, we prove the convergence of the bargaining protocol to a point in the core of the robust game with probability 1. In Section IV, we report some numerical simulations to illustrate our theoretical study, and we conclude in Section V.

Notation. We view vectors as columns. For a vector x, we use x_i or $[x]_i$ to denote its ith coordinate component. For two vectors x and y, we use $x < y$ ($x \leq y$) to denote $x_i < y_i$ ($x_i \leq y_i$) for all coordinate indices i. We let x' denote the transpose of a vector x, and $\|x\|$ denote its Euclidean norm. An $n \times n$ matrix A is row-stochastic if the matrix

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has nonnegative entries a_{ij} and $\sum_{j=1}^{n} a_{ij} = 1$ for all $i = 1, \ldots, n$. For a matrix A, we use a_{ij} or $[A]_{ij}$ to denote its ijth entry. A matrix A is doubly stochastic if both A and its transpose A' are row-stochastic. Given two sets U and S, we write $U \subset S$ to denote that U is a proper subset of S. We use $|S|$ for the cardinality of a given finite set S.

We write $P_X[x]$ to denote the projection of a vector x on a set X, and we write $\mathrm{dist}(x, X)$ for the distance from x to X, i.e., $P_X[x] = \arg\min_{y \in X} \|x - y\|$ and $\mathrm{dist}(x, X) = \|x - P_X[x]\|$, respectively. Given a set X and a scalar $\lambda \in \mathbb{R}$, the set λX is defined by $\lambda X \triangleq \{\lambda x \mid x \in X\}$. Given two sets $X, Y \subseteq \mathbb{R}^n$, the set sum $X + Y$ is defined by $X + Y \triangleq \{x + y \mid x \in X,\ y \in Y\}$. Given a set N of players and a function $\eta : S \mapsto \mathbb{R}$ defined for each nonempty coalition $S \subseteq N$, we write $< N, \eta >$ to denote the transferable utility (TU) game with the players' set N and the characteristic function η. We let η_S be the value $\eta(S)$ of the characteristic function η associated with a nonempty coalition $S \subseteq N$. Given a TU game $< N, \eta >$, we use $C(\eta)$ to denote the core of the game,

$$C(\eta) = \left\{ x \,\middle|\, \sum_{i \in N} x_i = \eta_N,\ \sum_{i \in S} x_i \geq \eta_S \text{ for all } S \subset N \right\},$$

where S is always considered to be non-empty.

II. Dynamic TU Game and Robust Game

In this section, we formulate a robust dynamic TU game and introduce a bargaining protocol that the players implement to reach an agreement on their allocations. We also provide some preliminary results for the protocol.

A. Problem Formulation and Bargaining Process

Consider a set of players $N = \{1, \ldots, n\}$ and the set of all possible (nonempty) *coalitions* $S \subseteq N$ among them. Let $m = 2^n - 1$ be the *number of possible coalitions*. We assume that time is discrete and use $t = 0, 1, 2, \ldots$ to index the time.

We consider a dynamic TU game, denoted $< N, \{v(t)\} >$, where $\{v(t)\}$ is a sequence of characteristic functions. In this game, the players are involved in a sequence of instantaneous TU games whereby, at each time t, the *instantaneous TU game* is $< N, v(t) >$ with $v(t) \in \mathbb{R}^m$ for all $t \geq 0$. Further, we let $v_S(t)$ denote *the value assigned to a nonempty coalition* $S \subseteq N$ in the instantaneous game $< N, v(t) >$. Throughout the rest of the paper, we assume that $S \neq \emptyset$, i.e., *we do not consider empty coalitions*.

In what follows, we deal with dynamic TU games where each characteristic function $v(t)$ is a random vector with realizations restricted to some set of values. Specifically, we assume that the grand coalition value $v_N(t)$ is deterministic for every $t \geq 0$, while the values $v_S(t)$ of the other coalitions $S \subset N$ have a common upper bound. These conditions are formally stated in the following assumption.

Assumption 1: There exists $v^{\max} \in \mathbb{R}^m$ such that for all $t \geq 0$,

$$v_N(t) = v_N^{\max},$$

$$v_S(t) \leq v_S^{\max} \qquad \text{for all coalitions } S \subset N.$$

We refer to the game $< N, v^{\max} >$ as *robust game*. We assume that the robust game has a nonempty core.

Assumption 2: The core $C(v^{\max})$ is not empty.

An immediate consequence of Assumptions 1 and 2 is that the core $C(v(t))$ of the instantaneous game is always not empty. This follows from the fact that $C(v^{\max}) \subseteq C(\eta)$ for any η satisfying $\eta_N = v_N^{\max}$ and $\eta_S \leq v_S^{\max}$ for $S \subset N$, and the assumption that the core $C(v^{\max})$ is not empty.

We assume that each player i is *rational and efficient*. This translates to each player $i \in N$ choosing his allocation vector within the set of allocations satisfying value constraints of all coalitions that include player i. This set is referred to as the *bounding set of player* i. For a generic game $< N, \eta >$, it is given by

$$X_i(\eta) = \left\{ x \in \mathbb{R}^n \mid \sum_{j \in N} x_j = \eta_N,\ \sum_{j \in S} x_j \geq \eta_S \right.$$
$$\left. \text{for all } S \subset N \text{ s.t. } i \in S \right\}.$$

Note that each $X_i(\eta)$ is polyhedral.

In what follows, we find it convenient to represent the bounding sets and the core in alternative equivalent forms. For each coalition $S \subseteq N$, let $e_S \in \mathbb{R}^n$ be the incidence vector for S, i.e., the vector with the coordinates given by

$$[e_S]_i = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{else.} \end{cases}$$

Then, the bounding sets and the core are given by

$$X_i(\eta) = \{x \in \mathbb{R}^n \mid e_N' x = \eta_N,\ e_S' x \geq \eta_S \text{ for all } S \subset N \text{ with } i \in S\}, \tag{1}$$

$$C(\eta) = \{x \in \mathbb{R}^n \mid e_N' x = \eta_N,\ e_S' x \geq \eta_S \text{ for all } S \subset N\}. \tag{2}$$

Observe that the core $C(\eta)$ of the game $< N, \eta >$ is the intersection of the bounding sets $X_i(\eta)$ of the players, i.e.,

$$C(\eta) = \cap_{i=1}^{n} X_i(\eta). \tag{3}$$

We now discuss the bargaining protocol where repeatedly over time each player $i \in N$ proposes an allocation vector. The allocation vector proposed by player i at time t is denoted by $x^i(t) \in \mathbb{R}^n$, where the jth component $x_j^i(t)$ represents the amount that player i would allocate to player j. To simplify the notation in the dynamic game $< N, \{v(t)\} >$, we let $X_i(t)$ denote the bounding set of player i for the instantaneous game $< N, v(t) >$, i.e., for all $i \in N$ and $t \geq 0$,

$$X_i(t) = \{x \in \mathbb{R}^n \mid e_N' x = v_N(t),\ e_S' x \geq v_S(t) \text{ for all } S \subset N \text{ with } i \in S\}. \tag{4}$$

Now, we focus on players interactions in time. We assume that each player may observe the allocations of a subset of the other players at any time, which are termed as the *neighbors of the player*. The players and their neighbors at time t can be represented by a directed graph $\mathcal{G}(t) = (N, \mathcal{E}(t))$, with the vertex set N and the set $\mathcal{E}(t)$ of directed links. A link

$(i,j) \in \mathcal{E}(t)$ exists if player j is a neighbor of player i at time t. We always assume that $(i,i) \in \mathcal{E}(t)$ for all t, which is natural since every player i can always access its own allocation vector. We refer to graph $\mathcal{G}(t)$ as a *neighbor-graph* at time t. In the graph $\mathcal{G}(t)$, a player j is a neighbor of player i (i.e., $(i,j) \in \mathcal{E}(t)$) only if player i can observe the allocation vector of player j at time t.

Given the players' neighbor-graph $\mathcal{G}(t)$, each player i negotiates allocations by averaging his allocation and his neighbors' allocations. More precisely, at time t, the bargaining process for each player i involves the player's individual bounding set $X_i(t)$, its own allocation $x^i(t)$ and the observed allocations $x^j(t)$ of some of his neighbors j. Formally, we let $\mathcal{N}_i(t)$ be the set of neighbors of player i at time t (including himself), i.e.,

$$\mathcal{N}_i(t) = \{j \in N \mid (i,j) \in \mathcal{E}(t)\}.$$

With this notation, the bargaining process is given by:

$$x^i(t+1) = P_{X_i(t)}\left[\sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)x^j(t)\right] \; \forall i \in N,\, t \geq 0 \quad (5)$$

where $a_{ij}(t) \geq 0$ is a scalar weight that player i assigns to the proposed allocation $x^j(t)$ of player j and $P_{X_i(t)}[\cdot]$ is the projection onto the player i bounding set $X_i(t)$. The initial allocations $x^i(0)$, $i \in N$, are selected randomly and independently of $\{v(t)\}$.

The bargaining in (5) can be written more compactly by introducing zero weights for players j whose allocations are not available to player i at time t. Specifically by defining $a_{ij}(t) = 0$ for all $j \notin \mathcal{N}_i(t)$ and $t \geq 0$, we have the following equivalent representation of the bargaining protocol:

$$x^i(t+1) = P_{X_i(t)}\left[\sum_{j=1}^{n} a_{ij}(t)x^j(t)\right] \; \forall i \in N,\, t \geq 0. \quad (6)$$

Here, $a_{ij}(t) = 0$ for $j \notin \mathcal{N}_i(t)$ and $a_{ij}(t) \geq 0$ for $j \in \mathcal{N}_i(t)$.

We now discuss the specific assumptions on the weights $a_{ij}(t)$ and the players' neighbor-graph that we will rely on. We let $A(t)$ be the weight matrix with entries $a_{ij}(t)$. We will use the following assumption for the weight matrices.

Assumption 3: Each matrix $A(t)$ is doubly stochastic with positive diagonal, and there exists a scalar $\alpha > 0$ such that

$$a_{ij}(t) \geq \alpha \qquad \text{whenever} \quad a_{ij}(t) > 0.$$

In view of the construction of matrices $A(t)$, we see that $a_{ij}(t) \geq \alpha$ for $j = i$ and perhaps for some players j that are neighbors of player i. The requirement that the positive weights are uniformly bounded away from zero is imposed to ensure that the information from each player diffuses to his neighbors in the network persistently in time. The requirement on the doubly stochasticity of the weights is used to ensure that in a long run each player has equal influence on the limiting allocation vector.

It is natural to expect that the connectivity of the players' neighbor-graphs $\mathcal{G}(t) = (N, \mathcal{E}(t))$ impacts the bargaining process. At any time, the instantaneous graph $\mathcal{G}(t)$ need not be connected. However, for the proper behavior of the bargaining process, the union of the graphs $\mathcal{G}(t)$ over a period of time is assumed to be connected.

Assumption 4: There is an integer $Q \geq 1$ such that the graph $\left(N, \bigcup_{\tau=tQ}^{(t+1)Q-1} \mathcal{E}(\tau)\right)$ is strongly connected for every $t \geq 0$.

Assumptions 3 and 4 together guarantee that the players communicate sufficiently often to ensure that the information of each player is persistently diffused over the network in time to reach every other player. Under these assumptions, we will study the dynamic bargaining process in (6). We want to provide conditions under which the process converges to an allocation in the core of the robust game. Before this, we provide some preliminary results in the following section.

B. Preliminary Results

In this section we derive some preliminary results pertinent to the core of the robust game. We also provide some error bounds for polyhedral sets applicable to the players' bounding sets $X_i(t)$. We later use these results to establish the convergence of the bargaining process in (6).

In our analysis we often use the following relation that holds for the projection operation on a closed convex set $X \subseteq \mathbb{R}^n$: for any $w \in \mathbb{R}^n$ and any $x \in X$,

$$\|P_X[w] - x\|^2 \leq \|w - x\|^2 - \|P_X[w] - w\|^2. \quad (7)$$

This relation is known as a strictly non-expansive projection property (see [7], volume II, 12.1.13 Lemma, page 1120).

We next relate the distance $\mathrm{dist}(x, C(\eta))$ from a point x to the core $C(\eta)$ with the distances $\mathrm{dist}(x, X_i(\eta))$ from x to the bounding sets $X_i(\eta)$. To do so, we use the polyhedrality of the bounding sets $X_i(\eta)$ and the core $C(\eta)$, as given in (1) and (2) respectively, and a special relation for polyhedral sets. This special relation states that for a nonempty polyhedral set $\mathcal{P} = \{x \in \mathbb{R}^n \mid a_\ell' x \leq b_\ell,\ \ell = 1, \ldots, r\}$, there exists a scalar $c > 0$ such that

$$\mathrm{dist}(x, \mathcal{P}) \leq c \sum_{\ell=1}^{r} \mathrm{dist}(x, H_\ell) \qquad \text{for all } x \in \mathbb{R}^n, \quad (8)$$

where $H_\ell = \{x \in \mathbb{R}^n \mid a_\ell' x \leq b_\ell\}$ and the scalar c depends only on the vectors $a_\ell, \ell = 1, \ldots, r$. Relation (8) is known as *Hoffman bound*, as it has been established by Hoffman [12]

Aside from the Hoffman bound, in establishing the forthcoming Lemma 1, we also use the fact that the square distance from a point x to a closed convex set X contained in an affine set H is given by

$$\mathrm{dist}^2(x, X) = \|x - P_H[x]\|^2 + \mathrm{dist}^2(P_H[x], X). \quad (9)$$

Now, we are ready to present the result relating the values $\mathrm{dist}^2(x, C(\eta))$ and $\mathrm{dist}^2(x, X_i(\eta))$.

Lemma 1: Let $< N, \eta >$ be a TU game with a nonempty core $C(\eta)$. Then, there is a constant $\mu > 0$ such that

$$\mathrm{dist}^2(x, C(\eta)) \leq \mu \sum_{i=1}^{n} \mathrm{dist}^2(x, X_i(\eta)) \qquad \text{for all } x \in \mathbb{R}^n,$$

where μ depends on the collection of vectors $\{\tilde{e}_S \mid S \subset N,\ S \neq \emptyset\}$, where each $\tilde{e}_S$ is the projection of e_S on the hyperplane $H = \{x \in \mathbb{R}^n \mid e_N' x = \eta_N\}$.

Proof: Since the hyperplane H contains the core $C(\eta)$ (see (2)), by relation (9) we have for all $x \in \mathbb{R}^n$,

$$\mathrm{dist}^2(x, C(\eta)) = \|x - P_H[x]\|^2 + \mathrm{dist}^2(P_H[x], C(\eta)). \quad (10)$$

The point $P_H[x]$ and the core $C(\eta)$ lie in the hyperplane H (an $n-1$-dimensional affine set). By applying the Hoffman bound relative to the affine set H (cf. Eq. (8)), we obtain

$$\mathrm{dist}(P_H[x], C(\eta)) \leq c \sum_{S \subset N} \mathrm{dist}(P_H[x], H \cap H_S),$$

where $H_S = \{x \in \mathbb{R}^n \mid e_S' x \geq \eta_S\}$, while the constant c depends on the collection $\{\tilde{e}_S, S \subset N\}$ of projections of vectors e_S on the hyperplane H for $S \subset N$. Thus, it follows

$$\begin{aligned}\mathrm{dist}^2(P_H[x], C(\eta)) &\leq c^2 \left(\sum_{S \subset N} \mathrm{dist}(P_H[x], H \cap H_S) \right)^2 \\ &\leq c^2(m-1) \sum_{S \subset N} \mathrm{dist}^2(P_H[x], H \cap H_S), \quad (11)\end{aligned}$$

where m is the number of nonempty subsets of N, and the last inequality follows by $(\sum_{j=1}^{\ell} a_j)^2 \leq \ell \sum_{j=1}^{\ell} a_j^2$, which holds for any set of scalars $\{a_j, j = 1, \ldots, \ell\}$ with $\ell \geq 1$. From Eqs. (10) and (11), we obtain for all $x \in \mathbb{R}^n$,

$$\begin{aligned}\mathrm{dist}^2(x, C(\eta)) &\leq \|x - P_H[x]\|^2 \\ &\quad + c^2(m-1) \sum_{S \subset N} \mathrm{dist}^2(P_H[x], H \cap H_S) \\ &\leq c_1 \sum_{S \subset N} \big(\|x - P_H[x]\|^2 \\ &\quad + \mathrm{dist}^2(P_H[x], H \cap H_S) \big),\end{aligned}$$

where $c_1 = \max\{1,\ c^2(m-1)\}$. Since the set H is affine, by Eq. (9) we have $\|x - P_H[x]\|^2 + \mathrm{dist}^2(P_H[x], H \cap H_S) = \mathrm{dist}^2(x, H \cap H_S)$, implying that for all $x \in \mathbb{R}^n$,

$$\mathrm{dist}^2(x, C(\eta)) \leq c_1 \sum_{S \subset N} \mathrm{dist}^2(x, H \cap H_S).$$

From the preceding relation, it follows for all $x \in \mathbb{R}^n$,

$$\mathrm{dist}^2(x, C(\eta)) \leq c_1 \sum_{S \subset N} |S| \, \mathrm{dist}^2(x, H \cap H_S), \quad (12)$$

where $|S|$ is the cardinality of coalition S. Note that

$$\begin{aligned}\sum_{S \subset N} |S| \, \mathrm{dist}^2(x, H \cap H_S) &= \sum_{S \subset N} \sum_{i \in S} \mathrm{dist}^2(x, H \cap H_S) \\ &= \sum_{i=1}^{n} \left(\sum_{\{S \subset N \mid i \in S\}} \mathrm{dist}^2(x, H \cap H_S) \right). \quad (13)\end{aligned}$$

We also note that $X_i(\eta) \subset H \cap H_S$ for each $S \subset N$ with $i \in S$, which follows by the definition of H_S and relation (1). For any two closed convex sets $X, Y \subseteq \mathbb{R}^n$ such that $X \subset Y$, we have $\mathrm{dist}(x, Y) \leq \mathrm{dist}(x, X)$ for any $x \in \mathbb{R}^n$. Thus, since $X_i(\eta) \subset H \cap H_S$ for each S with $i \in S$, it follows that for all $x \in \mathbb{R}^n$,

$$\mathrm{dist}(x, H \cap H_S) \leq \mathrm{dist}(x, X_i(\eta)). \quad (14)$$

By combining Eqs. (12)–(14) we obtain for all $x \in \mathbb{R}^n$,

$$\begin{aligned}\mathrm{dist}^2(x, C(\eta)) &\leq c_1 \sum_{i=1}^{n} \left(\sum_{\{S \subset N \mid i \in S\}} \mathrm{dist}^2(x, X_i(\eta)) \right) \\ &= c_1 \kappa \sum_{i=1}^{n} \mathrm{dist}^2(x, X_i(\eta)),\end{aligned}$$

where κ is the number of coalitions S that contain player i, which is the same number for every player (κ does not depend on i). The desired relation follows by letting $\mu = c_1 \kappa$, and by recalling that $c_1 = \max\{1, c^2(m-1)\}$ and that c depends on the projections $\tilde{e}_S$ of vectors $e_S, S \subset N$, on the hyperplane H. ■

Note that the scalar μ in Lemma 1 does not depend on the coalitions' values η_S for $S \neq N$. It depends only on the vectors e_S, $S \subseteq N$, and the grand coalition value η_N.

As a direct consequence of Lemma 1, we have the following result for the instantaneous game $< N, v(t) >$ under the assumptions of Section II-A.

Lemma 2: Let Assumptions 1 and 2 hold. We then have for all $x \in \mathbb{R}^n$ and all $t \geq 0$,

$$\mathrm{dist}^2(x, C(v(t))) \leq \mu \sum_{i=1}^{n} \mathrm{dist}^2(x, X_i(t)),$$

where $C(v(t))$ is the core of the game $< N, v(t) >$, $X_i(t)$ is the bounding set of player i, and μ is the constant from Lemma 1.

Proof: By Assumption 2, we have that the core $C(v^{\max})$ is nonempty. Furthermore, under Assumption 1, we have $C(v^{\max}) \subseteq C(v(t))$ for all $t \geq 0$, implying that the core $C(v(t))$ is nonempty for all $t \geq 0$.

Under Assumption 1, each core $C(v(t))$ is defined by the same affine equality corresponding to the grand coalition value, $e_N' x = v_N^{\max}$. Moreover, each core $C(v(t))$ is defined through the set of hyperplanes $H_S(t) = \{x \in \mathbb{R}^n \mid e_S' x \geq v_S(t)\}$, $S \subset N$, which have time invariant normal vectors e_S, $S \subseteq N$. Thus, the result follows from Lemma 1. ■

III. Convergence to Core of Robust Game

In this section, we prove convergence of the bargaining process in (6) to a random allocation in the core of the robust game with probability 1. We find it convenient to re-write bargaining protocol (6) by isolating a linear and a non-linear term. The linear term is the vector $w^i(t)$ defined as:

$$w^i(t) = \sum_{j=1}^{n} a_{ij}(t) x^j(t) \qquad \text{for all } i \in N \text{ and } t \geq 0. \quad (15)$$

Note that $w^i(t)$ is linear in players' allocations $x^j(t)$. The non-linear term is the error

$$e^i(t) = P_{X_i(t)}[w^i(t)] - w^i(t). \quad (16)$$

Using (15) and (16), we can rewrite protocol (6) as follows:

$$x^i(t+1) = w^i(t) + e^i(t) \qquad \text{for all } i \in N \text{ and } t \geq 0. \quad (17)$$

Recall that the weights $a_{ij}(t) \geq 0$ are such that $a_{ij}(t) = 0$ for all $j \notin \mathcal{N}_i(t)$. Also, recall that $A(t)$ is the matrix with entries $a_{ij}(t)$, which defines the vectors $w^i(t)$ in (15).

In what follows we will show that, with probability 1, bargaining protocol (15)–(17) converges to the core $C(v^{\max})$ of the robust game $< N, v^{\max} >$, provided that $v(t) = v^{\max}$ happens infinitely often in time with probability 1. To establish this we use some auxiliary results, which we develop in the next two lemmas.

The following lemma provides a result on the sequences $x^i(t)$ and shows that the errors $e^i(t)$ are diminishing.

Lemma 3: Let Assumptions 1 and 2 hold. Also, assume that each matrix $A(t)$ is doubly stochastic. Then, for bargaining protocol (15)–(17), we have

(a) $\left\{\sum_{i=1}^n \|x^i(t) - x\|^2\right\}$ converges for all $x \in C(v^{\max})$.
(b) $\sum_{t=0}^{\infty} \sum_{j=1}^n \|e^i(t)\|^2 < \infty$ and $\lim_{t\to\infty} \|e^i(t)\| = 0$ for all $i \in N$.

Proof: By $x^i(t+1) = P_{X_i(t)}[w^i(t)]$ and by strictly non-expansive property of the Euclidean projection on a closed convex set $X_i(t)$ (see (7)), we have for all $i \in N$, $t \geq 0$ and $x \in X_i(t)$,

$$\|x^i(t+1) - x\|^2 \leq \|w^i(t) - x\|^2 - \|e^i(t)\|^2. \quad (18)$$

Under Assumptions 1 and 2, the core $C(v^{\max})$ is contained in the core $C(v(t))$ for all $t \geq 0$, implying that $C(v^{\max}) \subseteq C(v(t))$ for all $t \geq 0$. Furthermore, since $C(v(t)) = \cap_{i=1}^n X_i(t)$, it follows that $C(v^{\max}) \subseteq X_i(t)$ for all $i \in N$ and $t \geq 0$. Therefore, relation (18) holds for all $x \in C(v^{\max})$. Thus, by summing the relations in (18) over $i \in N$, we obtain for all $t \geq 0$ and $x \in C(v^{\max})$,

$$\sum_{i=1}^n \|x^i(t+1) - x\|^2 \leq \sum_{i=1}^n \|w^i(t) - x\|^2 - \sum_{i=1}^n \|e^i(t)\|^2. \quad (19)$$

By the definition of $w^i(t)$ in (15), using the stochasticity of $A(t)$ and the convexity of the squared norm, we obtain

$$\sum_{i=1}^n \|w^i(t) - x\|^2 = \sum_{i=1}^n \left\| \sum_{j=1}^n a_{ij}(t) x^j(t) - x \right\|^2 \leq \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij}(t) \right) \|x^j(t) - x\|^2.$$

By the doubly stochasticity of $A(t)$, we have $\sum_{i=1}^n a_{ij}(t) = 1$ for every j, implying $\sum_{i=1}^n \|w^i(t) - x\|^2 \leq \sum_{i=1}^n \|x^i(t) - x\|^2$. By substituting this relation in (19), we arrive at

$$\sum_{i=1}^n \|x^i(t+1) - x\|^2 \leq \sum_{i=1}^n \|x^i(t) - x\|^2 - \sum_{i=1}^n \|e^i(t)\|^2. \quad (20)$$

The preceding relation shows that the scalar sequence $\{\sum_{i=1}^n \|x^i(t) - x\|^2\}$ is non-increasing for any given $x \in C(v^{\max})$. Therefore, the sequence must be convergent since it is nonnegative. Moreover, by summing the relations in (20) over $t = 0, \ldots, s$ and then, letting $s \to \infty$, we obtain

$$\sum_{t=0}^{\infty} \sum_{i=1}^n \|e^i(t)\|^2 \leq \sum_{i=1}^n \|x^i(0) - x\|^2,$$

which implies that $\lim_{t\to\infty} e^i(t) = 0$ for all $i \in N$. ■

In our next result, we will use the instantaneous average of players allocations, defined as follows:

$$y(t) = \frac{1}{n} \sum_{j=1}^n x^j(t) \qquad \text{for all } t \geq 0.$$

The result shows that the difference between the bargaining payoff vector $x^i(t)$ for any player i and the average $y(t)$ of these payoffs converges to 0 as time goes to infinity. The proof essentially uses the line of analysis that has been employed in [17], where the sets $X_i(t)$ are static, i.e., $X_i(t) = X_i$ for all t. In addition, we also use the rate result for doubly stochastic matrices, as established in [15].

Lemma 4: Let Assumptions 3 and 4 hold. Suppose that for the bargaining protocol (15)–(17) we have

$$\lim_{t\to\infty} \|e^i(t)\| = 0 \qquad \text{for all } i \in N.$$

Then, for every player $i \in N$ we have

$$\lim_{t\to\infty} \|x^i(t) - y(t)\| = 0, \qquad \lim_{t\to\infty} \|w^i(t) - y(t)\| = 0.$$

Proof: For any $t \geq s \geq 0$, define matrices

$$\Phi(t,s) = A(t)A(t-1)\cdots A(s+1)A(s),$$

with $\Phi(t,t) = A(t)$. Using the matrices $\Phi(t,s)$ and the expression for $x^i(t)$ in (17), we relate $x^i(t)$ with $x^i(s)$ at a time s for $0 \leq s \leq t-1$, as follows:

$$x^i(t) = \sum_{j=1}^n [\Phi(t-1,s)]_{ij}\, x^j(s) + \sum_{r=s+1}^{t-1} \left(\sum_{j=1}^n [\Phi(t-1,r)]_{ij}\, e^j(r-1) \right) + e^i(t-1). \quad (21)$$

Using the doubly stochasticity of $A(t)$, $y(t) = \frac{1}{n}\sum_{j=1}^n x^j(t)$ and relation (21), we obtain for all $t \geq s \geq 0$,

$$y(t) = \frac{1}{n} \sum_{j=1}^n x^j(s) + \frac{1}{n} \sum_{r=s+1}^{t} \left(\sum_{j=1}^n e^j(r-1) \right). \quad (22)$$

By our assumption, we have $\lim_{t\to\infty} \|e^i(t)\| = 0$ for all i. Thus, for any $\epsilon > 0$, there is an integer $\hat{s} \geq 0$ such that $\|e^i(t)\| \leq \epsilon$ for all $t \geq \hat{s}$ and all i. Using relations (21) and (22) with $s=\hat{s}$, we obtain for all i and $t \geq \hat{s}+1$,

$$\begin{aligned} \|x^i(t)-y(t)\| &= \Bigg\| \sum_{j=1}^n \Big([\Phi(t-1,\hat{s})]_{ij} - \frac{1}{n}\Big) x^j(\hat{s}) \\ &+ \sum_{r=\hat{s}+1}^{t-1} \sum_{j=1}^n \Big([\Phi(t-1,r)]_{ij} - \frac{1}{n}\Big) e^j(r-1) \\ &+ \Big(e^i(t-1) - \frac{1}{n}\sum_{j=1}^n e^j(t-1)\Big)\Bigg\| \\ &\leq \sum_{j=1}^n \Big|[\Phi(t-1,\hat{s})]_{ij} - \frac{1}{n}\Big| \, \|x^j(\hat{s})\| \\ &+ \sum_{r=\hat{s}+1}^{t-1} \sum_{j=1}^n \Big|[\Phi(t-1,r)]_{ij} - \frac{1}{n}\Big| \|e^j(r-1)\| \\ &+ \|e^i(t-1)\| + \frac{1}{n}\sum_{j=1}^n \|e^j(t-1)\|. \end{aligned}$$

Since $\|e^i(t)\| \leq \epsilon$ for all $t \geq \hat{s}$ and all i, it follows that

$$\begin{aligned} \|x^i(t)-y(t)\| \leq \sum_{j=1}^n \Big|[\Phi(t-1,\hat{s})]_{ij} - \frac{1}{n}\Big| \, \|x^j(\hat{s})\| \\ +\epsilon \sum_{r=\hat{s}+1}^{t-1} \sum_{j=1}^n \Big|[\Phi(t-1,r)]_{ij} - \frac{1}{n}\Big| \; + 2\epsilon. \end{aligned}$$

Under Assumptions 3 and 4, the following result holds for the matrices $\Phi(t,s)$, as shown in [14] (see there Corollary 1):

$$\Big|[\Phi(t,s)]_{ij} - \frac{1}{n}\Big| \leq \Big(1 - \frac{\alpha}{4n^2}\Big)^{\lceil \frac{t-s+1}{Q} \rceil - 2} \text{ for all } t \geq s \geq 0.$$

Substituting the preceding estimate in the estimate for $\|x^i(t)-y(t)\|$, we obtain

$$\begin{aligned} \|x^i(t)-y(t)\| \leq \Big(1 - \frac{\alpha}{4n^2}\Big)^{\lceil \frac{t-\hat{s}}{Q} \rceil - 2} \sum_{j=1}^n \|x^j(\hat{s})\| \\ +n\epsilon \sum_{r=\hat{s}+1}^{t-1} \Big(1 - \frac{\alpha}{4n^2}\Big)^{\lceil \frac{t-r}{Q} \rceil - 2} + 2\epsilon. \end{aligned}$$

Letting $t \to \infty$, we see that

$$\limsup_{t\to\infty} \|x^i(t)-y(t)\| \leq n\epsilon \sum_{r=\hat{s}+1}^{\infty} \Big(1 - \frac{\alpha}{4n^2}\Big)^{\lceil \frac{t-r}{Q} \rceil - 2} + 2\epsilon.$$

Note that $\sum_{r=\hat{s}+1}^{\infty} \left(1 - \frac{\alpha}{4n^2}\right)^{\lceil \frac{t-r}{Q} \rceil - 2} < \infty$, which by the arbitrary choice of ϵ yields

$$\lim_{t\to\infty} \|x^i(t)-y(t)\| = 0 \qquad \text{for all } i \in N.$$

Now, we focus on $\sum_{i=1}^n \|w^i(t)-y(t)\|$. Since $w^i(t) = \sum_{j=1}^n a_{ij}(t) x^j(t)$ and since $A(t)$ is stochastic, it follows

$$\sum_{i=1}^n \|w^i(t)-y(t)\| \leq \sum_{i=1}^n \sum_{j=1}^n a_{ij}(t) \|x^j(t)-y(t)\|.$$

Exchanging the order of the summations over, and then using the doubly stochasticity of $A(t)$, we have

$$\begin{aligned} \sum_{i=1}^n \|w^i(t)-y(t)\| &\leq \sum_{j=1}^n \Big(\sum_{i=1}^n a_{ij}(t)\Big) \|x^j(t)-y(t)\| \\ &= \sum_{j=1}^n \|x^j(t)-y(t)\|. \end{aligned}$$

Since $\lim_{t\to\infty} \|x^j(t)-y(t)\| = 0$ for all j, we have $\sum_{i=1}^n \|w^i(t)-y(t)\| \to 0$, implying $\|w^i(t)-y(t)\| \to 0$ for all i. ∎

So far, the polyhedrality of the sets $X_i(t)$ has not been used. We now combine all pieces together, namely Lemma 2 that exploits the polyhedrality of the bounding sets $X_i(t)$, Lemma 3 and Lemma 4. This brings us to the following convergence result for the robust game $< N, v^{\max} >$.

Theorem 1: Let Assumptions 1–4 hold. Also, assume that

$$\text{Prob}\,\{v(t) = v^{\max} \;\; i.o.\} = 1,$$

where $i.o.$ stands for infinitely often. Then, the players allocations $x^i(t)$ generated by bargaining protocol (15)–(17) converge with probability 1 to an allocation in the core $C(v^{\max})$, i.e., there is a random vector $\tilde{x} \in C(v^{\max})$ such that with probability 1,

$$\lim_{t\to\infty} \|x^i(t) - \tilde{x}\| = 0 \qquad \text{for all } i \in N.$$

Proof: By Lemma 3, the sequence $\{\sum_{i=1}^n \|x^i(t)-x\|^2\}$ is convergent for every $x \in C(v^{\max})$ and the errors $e^i(t)$ are diminishing for each player i, i.e., $\|e^i(t)\| \to 0$. Then, by Lemma 4 we have $\|x^i(t)-y(t)\| \to 0$ for all i. Hence,

$$\{\|y(t)-x\|\} \text{ is convergent for every } x \in C(v^{\max}). \quad (23)$$

We want to show that $\{y(t)\}$ is convergent and that its limit is in the core $C(v^{\max})$ with probability 1. For this, we note that since $x^i(t+1) \in X_i(t)$, it holds for all $t \geq 0$,

$$\sum_{i=1}^n \text{dist}^2\left(y(t+1), X_i(t)\right) \leq \sum_{i=1}^n \|y(t+1) - x^i(t+1)\|^2.$$

The preceding relation and $\|x^i(t)-y(t)\| \to 0$ for all $i \in N$ (cf. Lemma 4) imply

$$\lim_{t\to\infty} \sum_{i=1}^n \text{dist}^2\left(y(t+1), X_i(t)\right) = 0.$$

By Assumptions 1 and 2, and Lemma 2 we have for $t \geq 0$,

$$\text{dist}^2\left(y(t+1), C(v(t))\right) \leq \mu \sum_{i=1}^n \text{dist}^2\left(y(t+1), X_i(t)\right).$$

By combining the preceding two relations we see that

$$\lim_{t\to\infty} \text{dist}^2\left(y(t+1), C(v(t))\right) = 0. \quad (24)$$

By our assumption, we have that the event $\{v(t) = v^{\max}$ infinitely often$\}$ happens with probability 1. We now fix a realization $\{v_\omega(t)\}$ of the sequence $\{v(t)\}$ such that

$v_\omega(t) = v^{\max}$ holds infinitely often (for infinitely many t's). Let $\{t_k\}$ be a sequence such that

$$v_\omega(t_k) = v^{\max} \qquad \text{for all } k \geq 0.$$

All the variables corresponding to the realization $\{v_\omega(t)\}$ are denoted by a subscript ω. By relation (23) the sequence $\{y_\omega(t)\}$ is bounded, therefore $\{y_\omega(t_k)\}$ is bounded. Without loss of generality (by passing to a subsequence of $\{t_k\}$ if necessary), we assume that $\{y_\omega(t_k)\}$ converges to some vector $\tilde{y}_\omega$, i.e.,

$$\lim_{k\to\infty} y_\omega(t_k) = \tilde{y}_\omega.$$

Thus, the preceding two relations and Eq. (24) imply that $\tilde{y}_\omega \in C(v^{\max})$. Then, by relation (23), we have that $\{\|y_\omega(t) - \tilde{y}_\omega\|\}$ is convergent, from which we conclude that $\tilde{y}_\omega$ must be the unique accumulation point of the sequence $\{y_\omega(t)\}$, i.e.,

$$\lim_{t\to\infty} y_\omega(t) = \tilde{y}_\omega, \qquad \tilde{y}_\omega \in C(v^{\max}).$$

This and the assumption $\text{Prob}\,\{v(t) = v^{\max} \;\; i.o.\} = 1$, imply that the sequence $\{y(t)\}$ converges with probability 1 to a random point $\tilde{y} \in C(v^{\max})$. Since by Lemma 4 we have $\|x^i(t) - y(t)\| \to 0$ for every i, it follows that the sequences $\{x^i(t)\}, i = 1, \ldots, n$, converge with probability 1 to a common random point in the core $C(v^{\max})$. ■

IV. Numerical Illustrations

In this section, we report some numerical simulations. We consider a dynamic coalitional TU game with 3 players, so the number of possible nonempty coalitions is $m = 7$. The characteristic functions $v_S(t)$ are generated independently with identical uniform distribution over an interval. Specifically, at each time t, the value $v_{\{1\}}(t)$ is chosen randomly in the interval $[4, 7]$ with uniform probability independently of the other times. Similarly, the values $v_{\{2\}}(t)$ are generated in the interval $[0, 3]$. The grand coalition value is fixed to 10 at all times, and the other coalition values are 0.

We run 50 different Monte Carlo trajectories each one having 100 iterations. The number of iterations is chosen long enough to show the convergence of the protocols. All plots include the sampled average and sampled variance for the 50 different trajectories that were simulated. Each trajectory is generated by starting with the same initial allocations, which are given by $x^1(0) = [10\ 0\ 0]'$, $x^2(0) = [0\ 10\ 0]'$, and $x^3(0) = [0\ 0\ 10]'$. The sampled average is computed for each time $t = 1, \ldots, 100$, by fixing the time t and computing the average value of the 50 trajectory sample values for that time. The sampled variance is computed as the variance of the samples with respect to their sampled average.

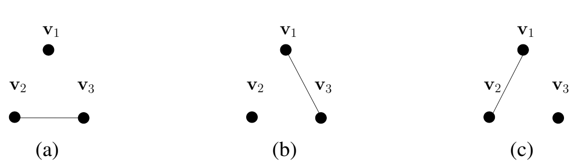

Fig. 1. Topology of players' neighbor-graph at three distinct times $t = 0$, 1 and 2.

Regarding the players' neighbor-graphs, we assume that the graphs are deterministic but time-varying. The graphs for the times $t = 0, 1, 2$ are as follows: player 2 and 3 connected at time $t = 0$ (see Figure 1(a)), then player 3 and 1 connected at time $t = 1$ (Figure 1(b)), and finally player 1 and 2 connected at time $t = 3$ (Figure 1(c)). These graphs are then repeated consecutively in the same order. In this way, the players' neighbor-graph is connected every 3 time units (Assumption 4 is satisfied with $Q = 3$).

The matrices $A(0)$, $A(1)$ and $A(2)$ that we associate with these three graphs, are respectively given by:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

These matrices are also repeated in the same order for the rest of the time. Thus, at any time t, the matrix $A(t)$ is doubly stochastic, with positive diagonal, and every positive entry bounded below by $\frac{1}{2}$, so Assumption 3 is satisfied with $\alpha = \frac{1}{2}$. All simulations are carried out with MATLAB on an Intel(R) Core(TM)2 Duo, CPU P8400 at 2.27 GHz and a 3GB of RAM. The run time of each simulation is around 90 seconds.

The characteristic function $v^{\max}$ for the robust game is obtained by considering the highest possible coalition values which results in $v^{\max} = [7\,3\,0\,0\,0\,0\,10]'$. The resulting core of the robust game is given by

$$C(v^{\max}) = \{x \in \mathbb{R}^3 : x_1 \geq 7,\, x_2 \geq 3,\, x_3 \geq 0,\, x_1 + x_2 \geq 0, \\ x_1 + x_3 \geq 0,\, x_2 + x_3 \geq 0,\, x_1 + x_2 + x_3 = 10\}.$$

We note that this core contains a single point, namely $[7\,3\,0]'$.

To ensure that $v(t) = v^{\max}$ infinitely often, as required by Theorem 1 for the convergence of the protocol, we adopt the following randomization mechanism. At each time $t = 1, \ldots, 100$, we flip a coin and if the outcome is "head" (probability 1/2), the coalitions' values $v_{\{1\}}(t)$ and $v_{\{2\}}(t)$ are extracted from the intervals $[4, 7]$ and $[0, 3]$, respectively, with uniform probability independently of the other times. If the outcome of the coin flip is "tail", then we assume that the robust game realizes and take $v(t) = v^{\max}$.

We next present the results obtained by the Monte Carlo runs for the bargaining protocol in (15)–(17). An illustration of a typical run with the allocations generated in periods $t = 0, 1, 2, 3$ is shown below:

$$\begin{array}{ll} v(0) = [6.8\ 2.7 \ldots 10]' & x^1(0) = [10\ 0\ 0]' \\ v(1) = [7\ 3 \ldots 10]' & x^1(1) = [10\ 0\ 0]' \\ v(2) = [4.4\ 1.1 \ldots 10]' & x^1(2) = [5\ 2.5\ 2.5]' \\ v(3) = [7\ 3 \ldots 10]' & x^1(3) = [7\ 1.5\ 1.5]' \end{array}$$

$$\begin{array}{ll} x^2(0) = [0\ 10\ 0]' & x^3(0) = [0\ 0\ 10]' \\ x^2(1) = [0\ 5\ 5]' & x^3(1) = [0\ 5\ 5]' \\ x^2(2) = [0\ 5\ 5]' & x^3(2) = [5\ 2.5\ 2.5]' \\ x^2(3) = [2.5\ 3.75\ 3.75]' & x^3(3) = [5\ 2.5\ 2.5]'. \end{array}$$

Recall that the initial allocations of the players are $x^1(0) = [10\ 0\ 0]'$, $x^2(0) = [0\ 10\ 0]'$, and $x^3(0) = [0\ 0\ 10]'$. At time $t = 1$, bargaining involves player 2 and 3 who update

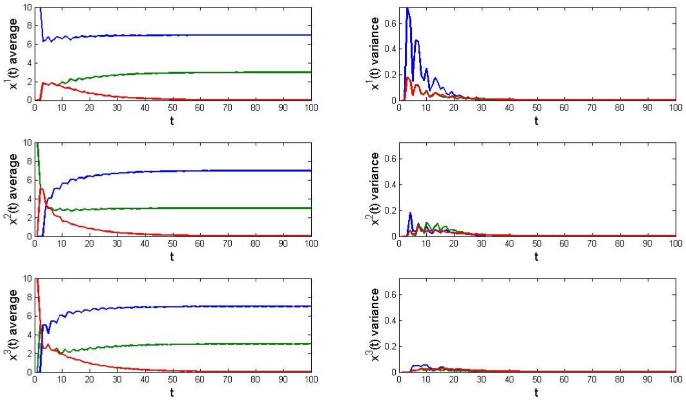

Fig. 2. Plots of the sampled average (left) and variance (right) of players' allocations $x^i(t)$, $i = 1, 2, 3$ generated by bargaining protocol (15)–(17). Sampled averages of the allocations $x^i(t)$ converge to the same point $\tilde{x} = [7\,3\,0]' \in C(v^{\max})$, while sampled variances go rapidly to zero.

the allocations respectively as $x^2(1) = [0\ 5\ 5]'$ and $x^3(1) = [0\ 5\ 5]'$. These allocations are feasible for their bounding sets so the projections on these sets are not performed. At time $t = 2$, the bargaining involves player 1 and 3 who update their allocations, respectively, as $x^1(2) = [5\ 2.5\ 2.5]'$ and $x^3(2) = [5\ 2.5\ 2.5]'$. Again, these allocations are feasible for their bounding sets and the projections are not performed. Finally, at time $t = 3$, the bargaining involves player 1 and 2 who update their allocations resulting in $x^1(3) = [7\ 1.5\ 1.5]'$ and $x^2(3) = [2.5\ 3.75\ 3.75]'$. Notice that $x^1(3)$ is obtained after player 1 projects onto his bounding set.

In Figure 2 we report our simulation results for the average of the sample trajectories obtained by Monte Carlo runs. We show the sampled average and variance of the allocations $x^i(t)$, $i = 1, 2, 3$ per iteration t. In accordance with the convergence result of Theorem 1, the sampled averages of the players' allocations $x^i(t)$ converge to the same point, namely $x = [7\,3\,0]'$ which is in the core of the robust game $C(v^{\max})$.

V. Conclusions

This article deals with dynamics and robustness within the framework of coalitional TU games. The novelty of the work lies in the design of a decentralized allocation process defined over a communication graph of players. The key properties that distinguish this work from the existing work on dynamic games are: (1) the introduction of a time-varying communication graph; and (2) the distributed bargaining protocol for players' allocations updates subject to local information exchange with neighboring players.

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