Games on Time-varying Networks

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Outline

Duopolistic Cournot Model (1838)

- Predates Nash equilibrium ideas
- Firms 1 and 2 produce items $q_1 \geq 0$ and $q_2 \geq 0$; let $Q = q_1 + q_2$
- Let $p(Q)$ be the price: $p(Q) = a - Q$, with $P(Q) = 0$ when $Q > a$
- Cost of production is $C_i(q_i) = cq_i$ with $c < a$ Normal form:
  - $S_i = [0, \infty)$ - strategy set; decision $q_i \geq 0$
  - Each firm wants to maximize its profit given the decision of the other firm

$$(\text{firm } i) \ : \max_{q_i \geq 0} \quad q_i(a - (q_i + q_{-i})) - cq_i \text{for } q_{-i} \text{ fixed}$$

Solution is a Nash equilibrium point (q_1^*, q_2^*):

$$q_1^* = \arg\max_{q_1 \geq 0} \quad q_1(a - (q_1 + q_2^*)) - cq_1$$
$$q_2^* = \arg\max_{q_2 \geq 0} \quad q_2(a - (q_1^* + q_2)) - cq_2$$

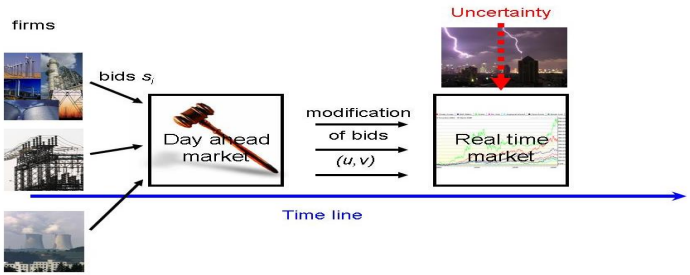
- Nash equilibrium in quantities:

$$q_1^* = \frac{1}{2}(a - q_2^* - c)$$
$$q_2^* = \frac{1}{2}(a - q_1^* - c)$$
$$q_1^* = q_2^* = \frac{a - c}{3}.$$

- Intuition: If $q_2^* = 0$, $q_1^* = \frac{a-c}{2}$, a monopolist level

Electricity markets

A 2-node Market



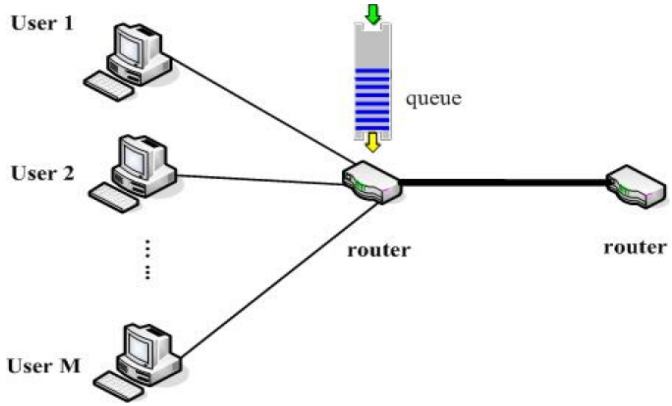
- Consider a 2-node electricity market. Suppose node 1 has a demand center while node 2 houses n electricity producers
- Producer i produces q_i MW (≥ 0) during a specific hour with capacity C_i
- Producer i's costs are $c_i q_i + \frac{1}{2} d_i q_i^2$.
- Transmission capacity of the link $(1, 2)$ is infinite
- Price of power at node 1 is given by $p(Q)$, $Q = \sum_{i=1}^{N} q_i$.
- Producers can affect prices and their payoffs are dependent on the generation levels of the remaining firms
- Specifically, producer i is faced by

$$\mathrm{G}_i(q_{-i}): \quad \begin{array}{ll} \text{maximize} & p(Q)q_i - c_i q_i - \frac{1}{2} d_i q_i^2 \\ \text{subject to} & q_i \leq C_i \\ & q_i \geq 0, \end{array}$$

  where $q_{-i} = (q_j)_{j \neq i}$.
- Nash equilibrium in generation levels: $(q_1^*, \ldots, q_n^*)$ where $q_i^* \in \arg\max_{0 \leq q_i \leq C_i} G_i(q^*{}_{-i})$ for all i.

Flow control games

- Multitude of applications on the internet
- A large number of users - inherently noncooperative in nature



- Demands for bandwidth lead to congestion
- Solution concept - a Nash equilibrium in flow rates, where each agent maximizes its utility less cost of sending flow

- User i has utility given by $U_i(x_i) := \ln(1 + x_i) + d_i,\ x_i \geq 0,$
- Cost of sending flow is $P_i(x_i, x_{-i}) := k_i x_i^2 \left(C - \sum_{i=1}^N x_i\right)^{-1}$
- Finally, flow decisions are required to stay feasible with respect to the capacity constraint $\sum_{i=1}^N x_i \leq C$.

Nash Equilibrium in flow decisions

- Users maximize the benefit of sending flow by solving

$$\begin{array}{lll} \mathrm{D}_i(x_{-i}) & \text{maximize} & U_i(x_i) - P_i(x_i; x_{-i}) \\ & \text{subject to} & x_i \geq 0,\ x_i + \sum_{j \neq i} x_j \leq C \end{array}$$

- Nash equilibrium in flow generation levels: $(x_1^*, \ldots, x_n^*)$ where $x_i^* \in \arg\max D_i(x_{i-}i, *)$, for all i and $\displaystyle\sum_{i=1}^N x_i^* \leq C$.

Continuous Games – Mixed-Strategy Nash Equilibrium

- In a host of practical settings, strategy decisions are continuous variables (as opposed to discrete such as in the bimatrix games).

Definition (Mixed-strategy Nash equilibria)

In the normal form game $G = \{S_1, \ldots, S_N; u_1, \ldots, u_N\}$, let $S_i = \{s_{i1}, \ldots, s_{iK_i}\}$. Then a mixed-strategy for player i is a distribution $p_i = (p_{i1}, \ldots, p_{iK_i})$, where $0 \leq p_{ik} \leq 1$ for all k and $\sum_{k=1}^{K_i} p_{ik} = 1$.

- A mixed strategy specifies the randomization over the set of pure strategies
- Mixed-strategies result in continuous decision variables (the probability masses p_i)

Continuous (Competitive) Game

- A number N of players (agents, users, nodes) indexed by $i = 1, \ldots, N$
- Each player i is controlling a decision variable $x_i \in \mathbb{R}^{n_i}$
- A player i is interested in minimizing its own cost $f_i(x_1, \ldots, x_N)$ in x_i subject to some (continuous) constraint set $K_i \subseteq \mathbb{R}^{n_i}$, given the decisions $\{x_j, j \neq i\}$ of other players:

(Player i problem)

$$\begin{array}{rl} \text{minimize} & f_i(x_1, \ldots, x_N) \\ \text{subject to} & x_i \in K_i \end{array}$$

- Game solutions: Nash equilibrium points $(x_1^*, \ldots, x_N^*)$ characterized by: for all $i = 1, \ldots, N$,

$$x_i^* \in \text{Arg} \min_{x_i \in K_i} f_i(x_1^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_N^*)$$

Outline

Variational Inequality Re-formulation

Let $x = (x_1, \ldots, x_N)$
For each i and a given x, let x_{-i} be the collection $(x_j)_{j \neq i}$
We will focus on convex games, where for every $i = 1, \ldots, N$:

- The functions $f_i(x_i; x_{-i})$ are convex and differentiable in x_i for every fixed x_{-i}
- The sets $K_i \subseteq \mathbb{R}^{n_i}$ are nonempty, closed and convex

Then, by using optimality conditions for Nash-equilibria we see that the Nash-equilibrium points are equivalently characterized by: for all $i = 1, \ldots, N$

$$\underbrace{\langle \nabla_{x_i} f_i(x^*), x_i - x_i^* \rangle \geq 0 \qquad \text{for all } x_i \in K_i,}_{\text{variational inequality}}$$

where $\nabla_{x_i} \phi(x)$ is the partial-gradient of a function ϕ with respect to the block-variable x_i

By stacking all inequalities we define

$$F(x) = \begin{bmatrix} \nabla_{x_1} f_1(x^*) \\ \nabla_{x_2} f_2(x^*) \\ \vdots \\ \nabla_{x_N} f_N(x^*) \end{bmatrix},$$

and the (Cartesian product) set

$$K = K_1 \times K_2 \times \cdots \times K_N.$$

With these, the Nash-equilibrium points x^* are characterized by the following (Cartesian) variational inequality

$$\langle F(x^*), x - x^* \rangle \geq 0 \qquad \text{for all } x \in K.$$

The variational inequality problem of determining a point $x^* \in K$ that satisfies the above relation is denoted by $\mathrm{VI}(K, F)$

Existence of Solutions

Consider VI(K, F) consisting of finding a point $x^* \in K$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0 \qquad \text{for all } x \in K.$$

Such an x^* is termed a (Nash equilibrium) solution of VI(K, F).

Theorem (Solution Existence/Uniqueness [FP03])

Let K be a compact set and let F be a continuous map. Then:

- *VI(K, F) has a solution.*
- *If F is monotone on K, i.e.,*

$$\langle F(x) - F(y), x - y \rangle \geq 0 \qquad \text{for all } x, y \in K.$$

Then, the solution is unique.

Solution Approaches

Consider VI(K, F) and assume that it has a solution $x^* \in K$.
It can be seen that $x^* \in K$ *solves VI*(K, F) *if and only if* x^* *is a fixed point* of a certain map, i.e.,

$$x^* = \Pi_K[x^* - \tau F(x^*)] \qquad \text{for any } \tau > 0,$$

where $\Pi_K[y]$ is the Euclidean projection of y on the set K:

$$\Pi_K[y] = \arg\min_{z \in K} \|z - y\|^2.$$

A class of iterative approaches is based on a fixed-point method for the map $y \mapsto \Pi_K[y - \tau(y)F(y)]$:

$$x^{k+1} = \Pi_K[x^k - \tau_k F(x^k)] \qquad \text{with } \tau_k > 0, \text{ starting with some } x^0 \in K.$$

Computations as seen by players

Our setting $F(x) = [\nabla_{x_1} f_1(x), \nabla_{x_2} f_2(x), \cdots, \nabla_{x_N} f_N(x)]$ and $K = K_1 \times \cdots \times K_N$.
Due to the structure of F and K, we have

$$\begin{aligned} & x^{k+1} = \Pi_K[x^k - \tau_k F(x^k)] \\ \iff \quad & x_i^{k+1} = \Pi_{K_i}[x_i^k - \tau_k \nabla_{x_i} f_i(x^k)] \text{ for } i = 1, \ldots, N. \end{aligned}$$

The later is also known as *gradient play*, as the players adjust their decisions based on gradient information of their private cost functions
Another popular approach is *best response play*, whereby players take turns in updating their decisions and choose the best decision given what the other players have chosen
At time k agent i is updating according to "best response", as follows:

$$x_i^{k+1} \in \text{Arg} \min_{x_i \in K_i} f_i(x_1^k, \ldots, x_{i-1}^k, x_i, x_{i+1}^k, \ldots, x_N^k).$$

The gradient play is natural where the costs are complicated (engineered systems).

Gradient Play: Computations distributed across players?

$$x_i^{k+1} = \Pi_{K_i}[x_i^k - \tau_k \nabla_{x_i} f_i(x^k)] \text{ for } i = 1, \ldots, N$$

Computations are "coupled" as each evaluation of $\nabla_{x_i} f_i(x^k)$ requires access to all players decisions $x^k = (x_1^k, \ldots, x_N^k)$.

- Most of the existing literature on VI's is not concerned with "distributed" aspect
- In some special instances, such as Electricity Markets and Network Flow Games [Sri04, SS07, KNS09, KNS11], an entity collects decisions x_i^k from all players and returns "prices" as a mechanism to guide players actions toward a Nash equilibrium
- Another special class of "aggregative games" is discussed next

Outline

An Aggregative Game

We have a system with N agents, where each agent i solves the following problem:

$$\begin{aligned} \text{minimize} \quad & f_i(x_i, \bar{x}), \quad \bar{x} = \sum_{j=1}^{N} x_j \\ \text{subject to} \quad & x_i \in K_i \subseteq \mathbb{R}^n. \end{aligned} \tag{AggGame}$$

- $f_i(\cdot,\cdot)$ and K_i are known only to agent i
- Agent i decides on x_i, and does not have access to the decisions x_j of the other agents
- Unlike [Jensen06, MartimortStole2010], the emphasis here is on computation of an equilibrium in the absence of instantaneous access to the aggregate value $\bar{x}$
- This will be achieved by allowing agents to communicate locally over a connected network

Motivating Example

Example (Networked Nash-Cournot Oligopoly)

- N firms competing at M locations
- Optimization problem for firm i

$$\begin{aligned} \text{minimize} \quad & \sum_{\ell=1}^{M} \left(c_{i\ell}(x_{i\ell}) - p_\ell(\bar{s}_\ell)s_{i\ell}\right) + t_i(x_{i1},\ldots,x_{iM}) \\ \text{subject to} \quad & \sum_{\ell=1}^{M} x_{i\ell} = \sum_{\ell=1}^{M} s_{i\ell}, \\ & x_{i\ell}, s_{i\ell} \geq 0, \quad x_{i\ell} \leq \text{cap}_{i\ell}, \qquad \ell = 1,\ldots,M. \end{aligned}$$

- $x_{i\ell}$ and $s_{i\ell}$ are production and sales for firm i at location ℓ
- p_ℓ denotes the price function and $\bar{s}_\ell = \sum_{j=1}^{N} s_{j\ell}$
- $p_\ell(\bar{s}_\ell)s_{i\ell}$ is the revenue at location ℓ for firm i
- t_i is the transportation cost

Computational algorithm

$$\begin{aligned} \text{minimize} \quad & f_i(x_i, \bar{x}), \quad \bar{x} = \sum_{j=1}^{N} x_j \\ \text{subject to} \quad & x_i \in K_i \subseteq \mathbb{R}^n. \end{aligned} \tag{AggGame}$$

- Are there distributed algorithms ?? [1]
- Under ideal conditions and suitable assumption

$$x_i^{k+1} = \Pi_{K_i}[x_i^k - \alpha \nabla_{x_i} f_i(x_i^k, \ \bar{x}^k \ )]$$

- What is the challenge then?

[1] Projection based [Facchinei-Pang03, Alpcan03] and their regularized variants [Kannan-Shanbhag10]

Computational algorithm

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Goal :

- Develop distributed algorithms that can aid agents to learn the aggregate and lead to an equilibrium point
- And still satisfy the informational constraints

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- By allowing agents to communicate their beliefs of the aggregate: $(\bar{x} = \sum_{i=1}^{N} x_i)$

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How?

- By allowing agents to communicate their beliefs of the aggregate: $(\bar{x} = \sum_{i=1}^{N} x_i)$
- But not their decisions: (x_i)
- This is in sharp contrast with "distributed optimization" setting (since here agents are competing)

Assumptions for a Unique Equilibrium

Basic Assumption 1

For each $i = 1, \ldots, N$,

- The set $K_i \subset \mathbb{R}^n$ is compact and convex
- Function $f_i(x_i, y)$ is continuously differentiable in (x_i, y) over some open set containing the set $K_i \times \bar{K}$, where $\bar{K} = \sum_{i=1}^{N} K_i$
- The function $x_i \mapsto f_i(x_i, s(x))$ is convex over the set K_i.

- These assumptions are sufficient for the existence of an equilibrium.

Additional Assumption: Uniqueness

- Define mapping $\Phi(x)$ by

$$\Phi(x) \triangleq \begin{pmatrix} F_1(x_1, s(x)) \\ \vdots \\ F_N(x_N, s(x)) \end{pmatrix}$$

$$F_i(x_i, s(x)) = \nabla_{x_i} f_i(x_i, s(x)), \qquad s(x) = \sum_{j=1}^{N} x_j$$

Basic Assumption 2 (Strict Monotonicity)

The mapping $\Phi(x)$ is strictly monotone over $K_1 \times \cdots \times K_N$, i.e.,

$$(\Phi(x) - \Phi(x'))^T (x - x') > 0, \quad \forall x, x' \in K_1 \times \cdots \times K_N.$$

Equilibrium conditions

Unique Equilibrium

Consider the aggregative Nash game defined in (AggGame). Suppose Basic Assumptions hold. Then, the game admits a unique Nash equilibrium.

Further Assumptions

Basic Assumption 3 [Computational]

The mapping $F_i(x_i, u)$ is uniformly Lipschitz continuous in u over $\bar{K} = K_1 + \cdots + K_N$, for every fixed $x_i \in K_i$ i.e., for some $L_{-i} > 0$

$$\|F_i(x_i, u) - F_i(x_i, z)\| \leq L_{-i}\|u - z\|.$$

Outline

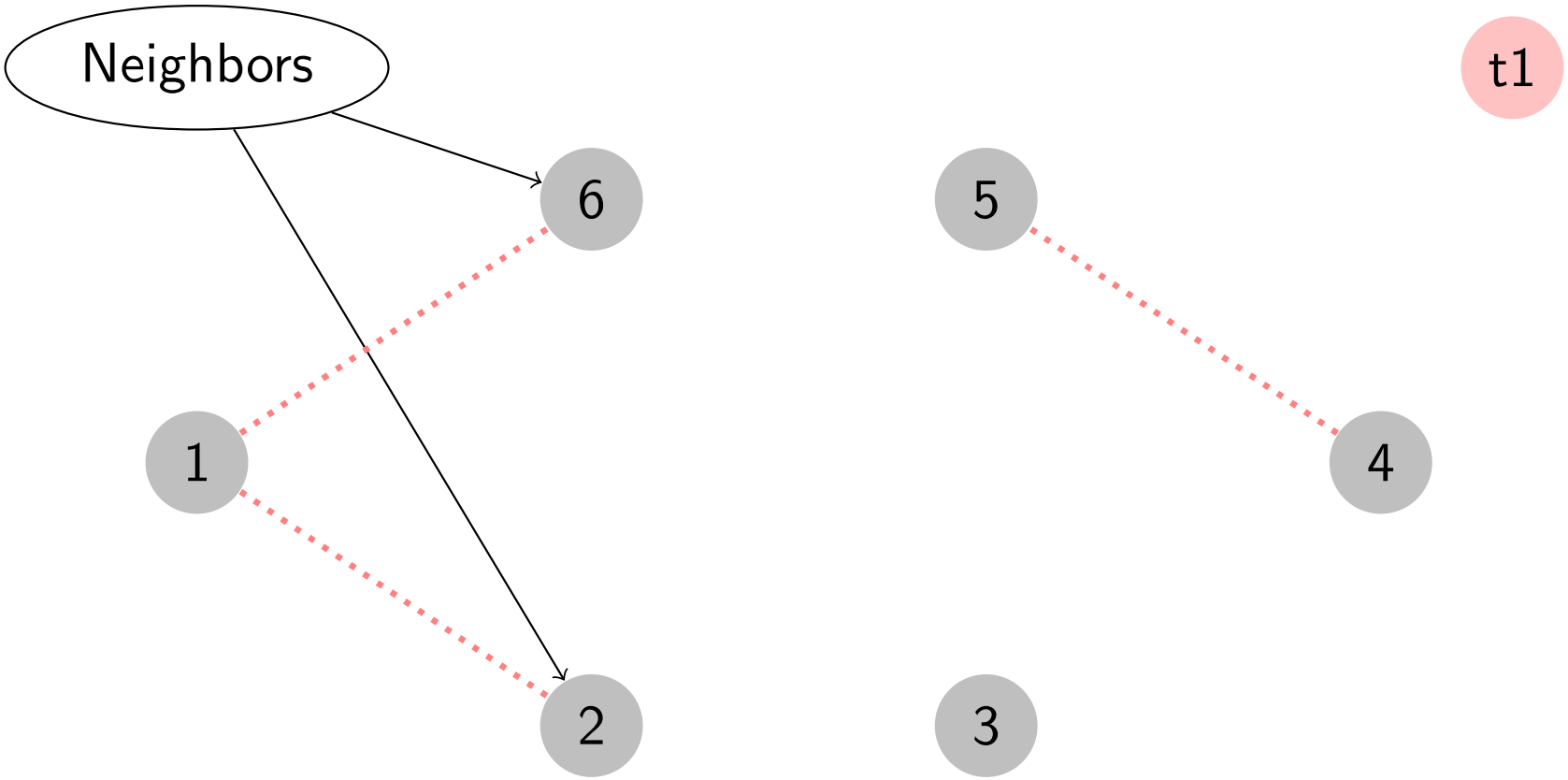
Synchronous setting

- A time varying undirected connectivity graph $\mathcal{G}_k = (\mathcal{N}, \mathcal{E}_k)$
- $\mathcal{N} = \{1, \ldots, N\}$ be a set of N agents
- $\mathcal{E}_k$ is the set of edges at time k
- $\mathcal{N}_i(k)$: immediate neighbors of agent i at time k

Connectivity Assumption

There exists an integer $Q \geq 1$ such that the graph $\bigcup_{l=1}^{Q} \mathcal{G}_{l+k}$ is connected for all k.

Limited connectivity of agents



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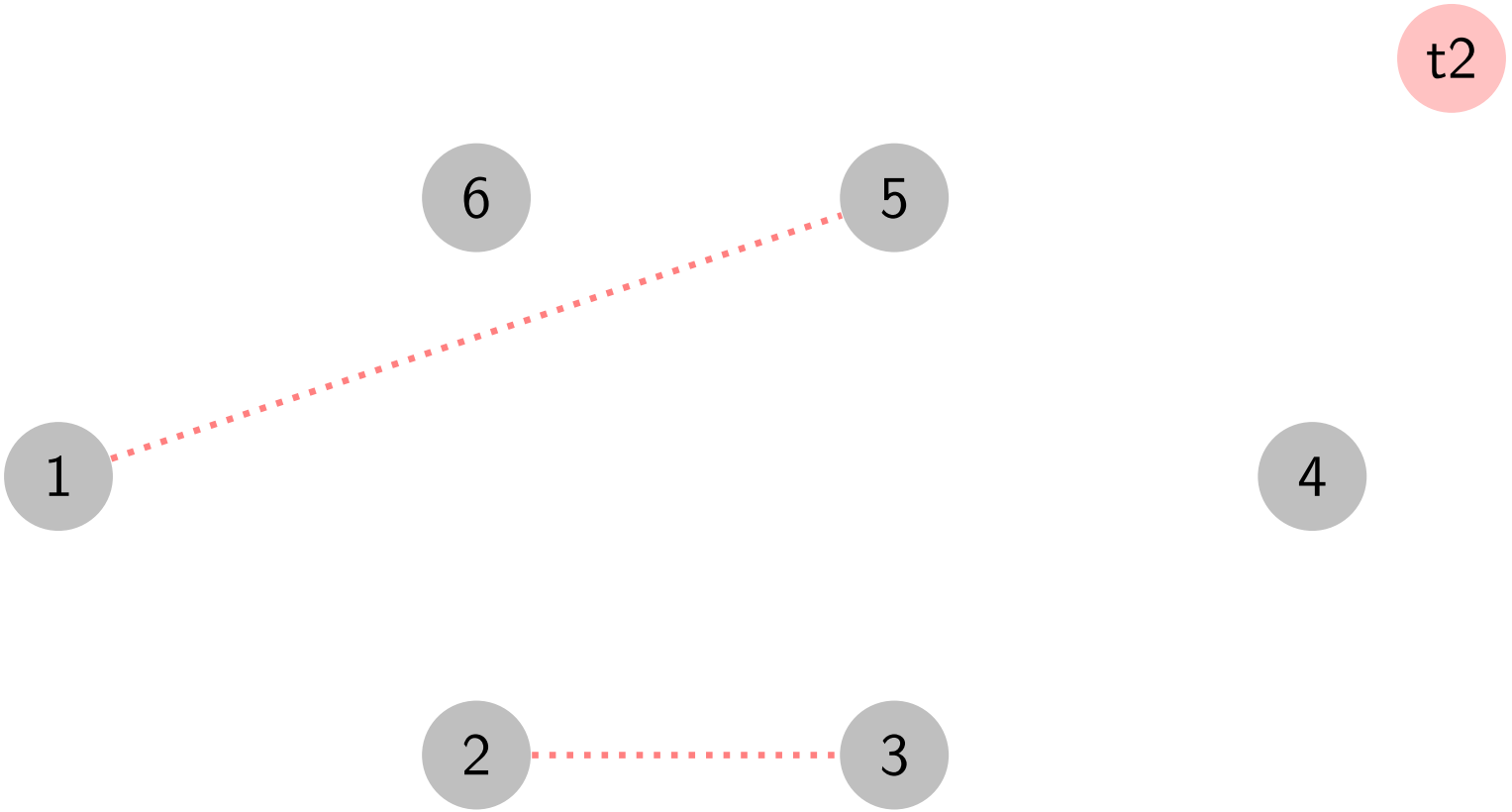

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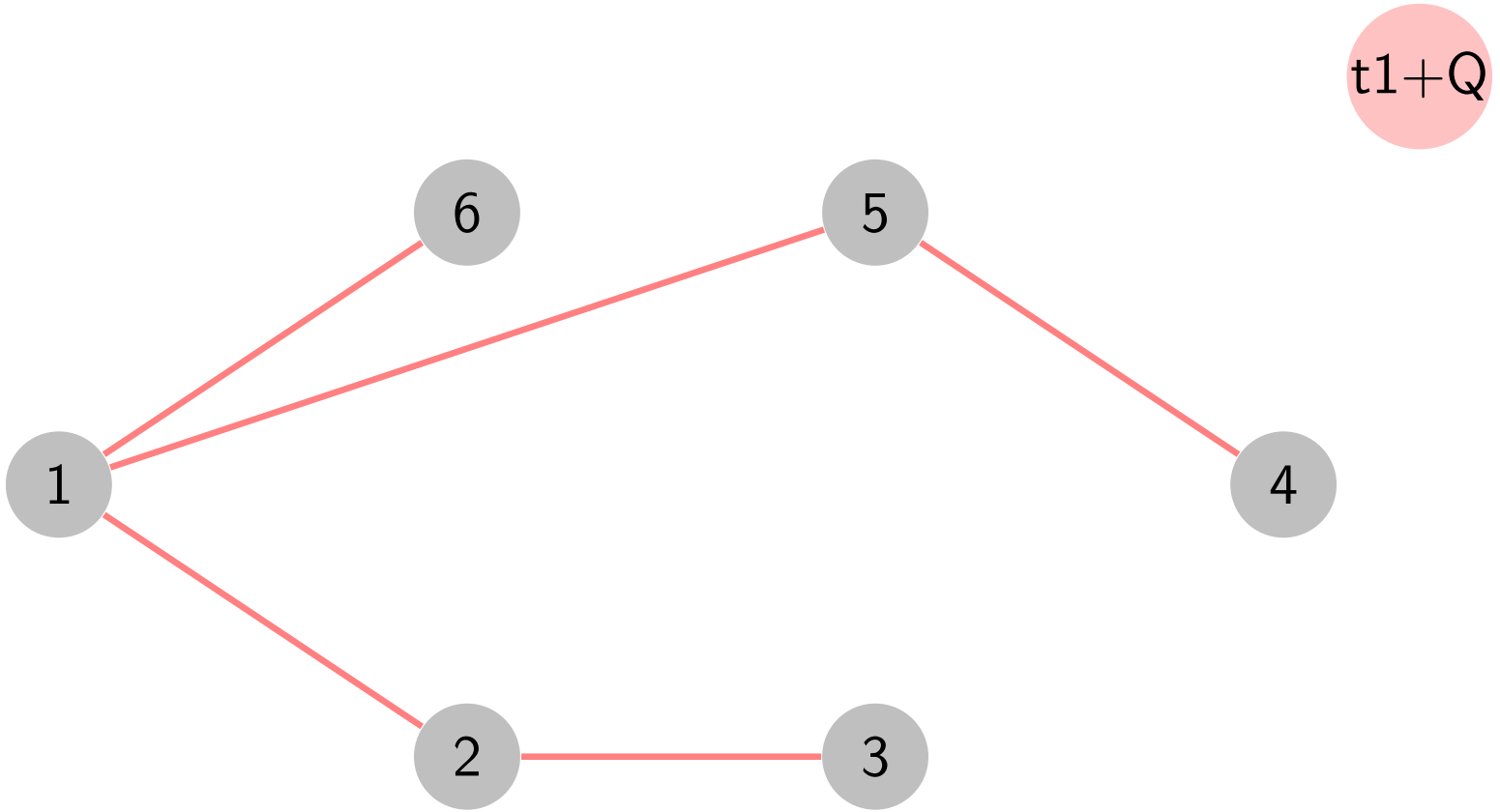

Variational Inequality Formulation

We want to determine the point $x^* = (x_1^*, \ldots, x_N^*) \in K$ such that

$$\langle \Phi(x^*), x - x^* \rangle \geq 0 \qquad \text{for all } x \in K,$$

where

$$\Phi(x) \triangleq \begin{pmatrix} F_1(x_1, s(x)) \\ \vdots \\ F_N(x_N, s(x)) \end{pmatrix}$$

$$F_i(x_i, s(x)) = \nabla_{x_i} f_i(x_i, s(x)), \qquad s(x) = \sum_{j=1}^{N} x_j$$

We use a suitable analog of

$$x_i^{k+1} = \Pi_{K_i}[x_i^k - \alpha \nabla_{x_i} f_i(x_i^k, s(x^k))$$

where we distribute learning of $s(x^k)$ in the network

Outline of the synchronous algorithm: the three steps

1. Learn aggregate:[2]
   - Each player maintains and updates an estimate v_i^k of $s(x^k) = \sum_{i=1}^N x_i^k$ (tricky since x_j^k's are players' private variables & are changing in time)
   - Every player communicates its estimate v_i^k to the neighbors
2. Update decision:
   - Using aligned estimate agents update their decision (x_i^k)
3. Update estimate of aggregate:
   - Agents update their estimate to reflect their current decision.

[2] Inspiration: Consensus based algorithm [Tsi84, SRNV12], Distributed optimization problem [NO09, SNS13]

Synchronous algorithm

1. Learn aggregate:

$$\hat{v}_i^k = \sum_{j \in \mathcal{N}_i(k)} w_{ij}(k) v_j^k, \quad v_i^0 = x_i^0,$$

$w_{ij}(k)$: agent i's weight to agent j's information at step k

2. Update decision:

$$x_i^{k+1} = \Pi_{K_i}[x_i^k - \alpha_k F_i(x_i^k, N\hat{v}_i^k)].$$

where agent i uses its estimate $N\hat{v}_i^k$ instead of $\sum_{j=1}^N x_j^k$

3. Update estimate of aggregate:

$$v_i^{k+1} = \hat{v}_i^k + x_i^{k+1} - x_i^k$$

Step suggested by Ram in [Ram-Nedić-Veeravali 2012]

Weight assumptions

Synchronous Weight

For all $i \in \mathcal{N}$ and all $k \geq 0$, the following hold:

(i) $w_{ij}(k) \geq \delta$ for all $j \in \mathcal{N}_i(k)$ and $w_{ij}(k) = 0$ for $j \notin \mathcal{N}_i(k)$;

(ii) $\sum_{j=1}^{N} w_{ij}(k) = 1$ for all i;

(iii) $\sum_{i=1}^{N} w_{ij}(k) = 1$ for all j.

Stepsize assumptions

Synchronous Stepsize

The stepsize α_k is chosen such that the following hold:

(i) The sequence $\{\alpha_k\}$ is monotonically non-increasing i.e., $\alpha_{k+1} \leq \alpha_k$ for all k;

(ii) $\sum_{k=0}^{\infty} \alpha_k = \infty$;

(iii) $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$.

Convergence of synchronous algorithm

Proposition ([KNS12a])

Let Basic Assumptions hold. Further let the Synchronous Weight and Synchronous Stepsize assumption hold. Then, the sequence $\{x^k\}$ generated by the Synchronous Algorithm converges to the (unique) solution x^* of the game.

Proof sketch

1. Gradient projection algorithm technique ($\|x^k - x^*\|$)
   - Utilizing the Lipschitz continuity of the map leads to a bound on $\|x_i^{k+1} - x_i^*\|$ in terms of $\|Nv_i^k - \bar{x}^*\|$
2. Estimate the errors for the beliefs ($\|Nv_i^k - \bar{x}^*\|$)
   - Introduce $y^k = \sum_{\ell=1} v_\ell^k / N$, the average estimate under perfect information
   - y^k also tracks the average of the aggregate i.e., $y^k = \sum_{\ell=1}^N x_\ell^k / N$
   - Construct bounds for $\|v_i^k - y^k\|$ and $\|Ny^k - \bar{x}^*\|$

Moving ahead

Convergence established for synchronous regime but
Key Drawback:

- Synchronization is a challenge in large networks
- Requires coordination in terms of stepsize

Remedy: Desynchronization

- No explicit dependency
- Allows independent solution of stepsizes
- Cannot accommodate time-varying graphs

Outline

Asynchronous setting

Connected Graph

The undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ is connected.

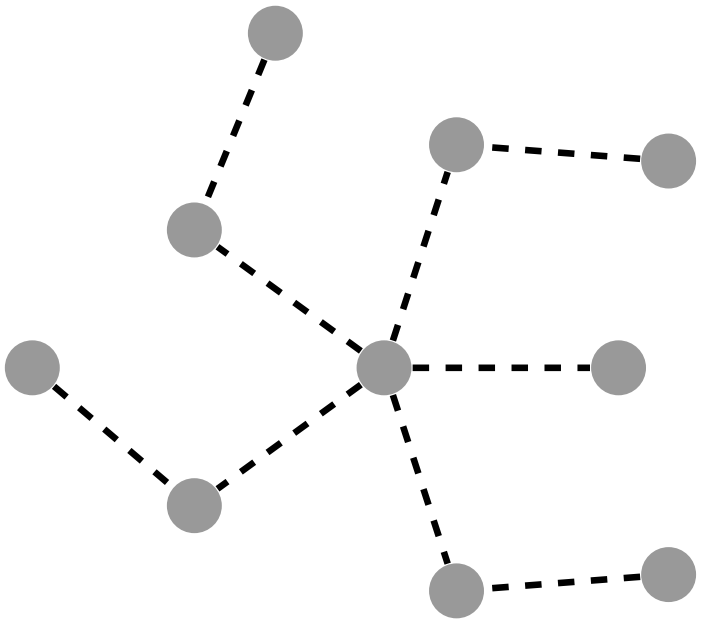

Figure: A depiction of a gossip communication.

Asynchronous setting

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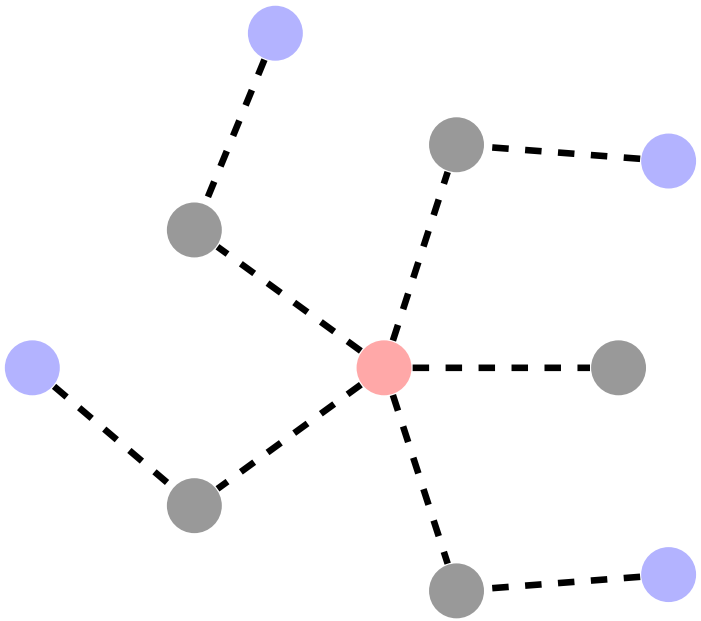

Figure: A depiction of a gossip communication.

Asynchronous setting

Connected Graph

The undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ is connected.

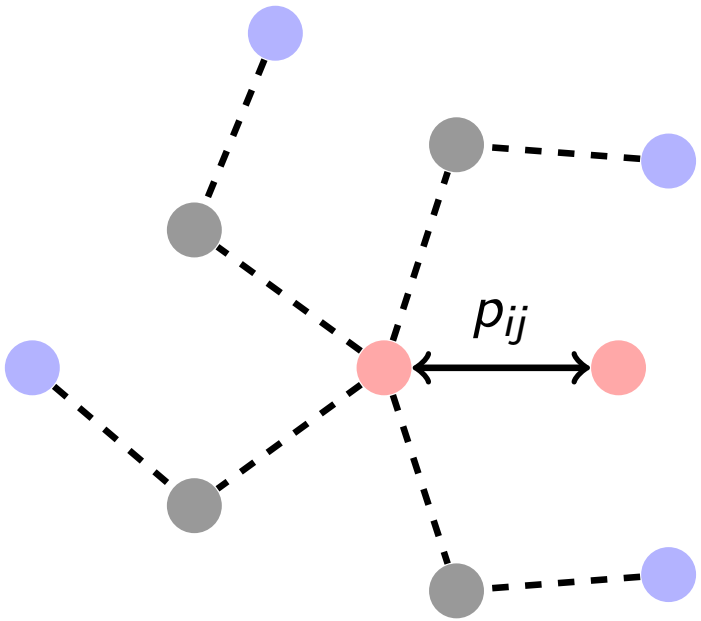

Figure: A depiction of a gossip communication.

Outline of the asynchronous algorithm: the four steps

1. Agents have local clocks ticking at rate 1
2. Wake Up:
   - Agent I^k wakes up at time k and then contacts J^k
3. Learn aggregate:
   - Only agents $\{I^k, J^k\}$ communicate their estimate (v_i^k) and perform mixing
4. Update decision:
   - Using aligned estimate agents update their decision (x_i^k)
5. Update estimate of aggregate:
   - Agents update their estimate to reflect their current decision.

Asynchronous algorithm

- Wake Up:
- Agent I^k wakes up at time k and then contacts J^k

1. Learn aggregate:

$$\hat{v}_i^k = \frac{v_{I^k}^k + v_{J^k}^k}{2} \quad \text{for } i \in \{I^k, J^k\}, \quad v_i^0 = x_i^0,$$

2. Update decision:

$$x_i^{k+1} = \left(\Pi_{K_i}[x_i^k - \alpha_{k,i} F_i(x_i^k, N\hat{v}_i^k)] - x_i^k\right) \mathbb{1}_{\{i \in \{I^k, J^k\}\}} + x_i^k$$

3. Update estimate of aggregate:

$$v_i^{k+1} = \hat{v}_i^k + x_i^{k+1} - x_i^k.$$

Stepsize Assumptions

Asynchronous Stepsize

The stepsize $\alpha_{k,i}$ is updated as follows:

$$\alpha_{k,i} = \frac{1}{\Gamma_k(i)}, \quad \text{where}$$

- $\Gamma_k(i)$ denotes the number of updates that agent i has executed up to time k inclusively.

Convergence of asynchronous algorithm

Proposition ([KNS12b])

Let Basic Assumption, Connected Graph and the Asynchronous Stepsize assumption hold. Then, the sequence $\{x^k\}$ generated by the Asynchronous Algorithm converges to the (unique) solution x^* of the game.

Convergence analysis of asynchronous algorithm

Key property: of some k large enough almost surely we have

$$\alpha_{k,i} \leq \frac{2}{kp_i}, \qquad \left|\alpha_{k,i} - \frac{1}{kp_i}\right| \leq \frac{2}{k^{3/2-q}p_{\min}^2}.$$

1. Stochastic gradient technique $\left(\mathbb{E}\left[\|x^k - x^*\|^2 \mid \mathcal{F}_{k-1}\right]\right)$
2. Estimate the expected errors for the beliefs $\left(\mathbb{E}\left[\|Nv_i^k - \bar{x}^*\| \mid \mathcal{F}_{k-1}\right]\right)$

Constant stepsize asynchronous algorithm

Agents employ a constant deterministic yet uncoordinated stepsize

- Wake Up:
- Agent I^k wakes up at time k and then contacts J^k

1. Learn aggregate:

$$\hat{v}_i^k = \frac{v_{I^k}^k + v_{J^k}^k}{2} \quad \text{for } i \in \{I^k, J^k\}, \quad v_i^0 = x_i^0,$$

2. Update decision:

$$x_i^{k+1} = \left(\Pi_{K_i}[x_i^k - \alpha F_i(x_i^k, N\hat{v}_i^k)] - x_i^k\right) \mathbb{1}_{\{i \in \{I^k, J^k\}\}} + x_i^k$$

3. Update estimate of aggregate:

$$v_i^{k+1} = \hat{v}_i^k + x_i^{k+1} - x_i^k.$$

Basic Assumption 4

The mapping $F_i(x_i, u)$ is uniformly Lipschitz continuous in x_i over K_i, for every fixed $u \in \bar{K}$ i.e., for some $L_i > 0$

$$\|F_i(x_i, u) - F_i(y_i, u)\| \leq L_i \|x_i - y_i\|;$$

Error bound: constant stepsize asynchronous algorithm

Proposition

Let Basic Assumptions hold with the mapping Φ be strongly monotone over the set K. Further let Connected Graph assumption hold. Then, the following holds for the sequence $\{x^k\}$ generated by the Asynchronous Algorithm with stepsize $\alpha_{k,i} = \alpha_i$

$$\limsup_{k\to\infty} \mathbb{E}[\|x^{k+1} - x^*\|^2] \leq Err(N, n, \mathcal{G}),$$

where x^* is the unique equilibrium of the game.

Expression for error

$$Err(N,n,\mathcal{G}) = \frac{4p_{\max}\alpha_{\max}^2 C^2 N + 2p_{\max}\alpha_{\max}^2 B\frac{\sqrt{2nN}\,C}{1-\sqrt{\lambda}}}{2\mu p_{\min}\alpha_{\min} - 2p_{\max}(\max_i L_i)(\alpha_{\max}-\alpha_{\min})}$$

- where $\alpha_{\max} = \max_{i=1,\ldots,N}\{\alpha_i\}$, $\alpha_{\min} = \min_{i=1,\ldots,N}\{\alpha_i\}$,
- $p_{\max} = \max_{i=1,\ldots,N}\{p_i\}$ and $p_{\min} = \min_{i=1,\ldots,N}\{p_i\}$.
- p_i is the probability of agent i updating
- $B = (\max_i L_{-i})NM$
- $M \geq \max_{x_i,z_i\in K_i}\|x_i - z_i\|$ for all i

Expression for error - equal step sizes

$$Err(N,n,\mathcal{G}) = \frac{\alpha p_{\max}}{\mu p_{\min}}\left(2C^2N + B\frac{\sqrt{2nN}\,C}{1-\sqrt{\lambda}}\right)$$

- where $\alpha_{\max} = \max_{i=1,\ldots,N}\{\alpha_i\}$, $\alpha_{\min} = \min_{i=1,\ldots,N}\{\alpha_i\}$,
- $p_{\max} = \max_{i=1,\ldots,N}\{p_i\}$ and $p_{\min} = \min_{i=1,\ldots,N}\{p_i\}$.
- p_i is the probability of agent i updating
- $B = (\max_i L_{-i})NM$
- $M \geq \max_{x_i,z_i\in K_i}\|x_i - z_i\|$ for all i

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Further extensions

- More general case can be solved

$$\begin{aligned} \text{minimize} \quad & f_i(x_i, \bar{x})), \quad \bar{x} = \sum_{j=1}^{N} x_j \\ \text{subject to} \quad & x_i \in K_i \subseteq \mathbb{R}^n, \\ & g(x)_1 \leq 0, \ldots, g_m(x) \leq 0 \end{aligned} \qquad \text{(AggGame)}$$

  under suitable assumptions on g_i.

- Push-sum can be used for mixing (instead of convex combinations)