Dynamic Coalitional TU Games: Distributed Bargaining among Players' Neighbors

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Introduction

- A tutorial paper

  Coalitional Game Theory for Communication Networks

  by W. Saad, Z. Han, M. Debbah, A. Hjorungnes, and T. Basar, 2009

Coalitional Transferable Utility (TU) Game

A TU game (N, v) is specified by

- Set $N = \{1, 2, \ldots, n\}$ of players
- Characteristic function v that assigns a scalar value v_S to every nonempty $S \subseteq N$.
  - Formally: we let $\mathscr{P}(N)$ be the set of all possible (nonempty) subsets of N and let m be its cardinality.
  - Then, the characteristic function is a vector in $\mathbb{R}^m$, $v : \mathscr{P}(N) \to \mathbb{R}^m$.
- Value v_S can be thought of as a monetary value that the players in S will distribute among themselves in some fair manner.
- The **grand coalition** N is stable when no player has incentive to leave coalition N, i.e., the core C of the game is nonempty

$$C = \{x \in \mathbb{R}^n \mid e_N' x = v_N, \quad e_S' x \geq v_S \text{ for } S \subset N\},$$

where x is an allocation vector for players and e_S is the incidence vector of coalition S:

$$[e_S]_i = \begin{cases} 1 & \text{when } i \in S, \\ 0 & \text{when } i \notin S. \end{cases}$$

- **Bargaining** is an allocation process by which the players reach an agreement on some allocation in the core.

Bargaining: determining an allocation in the core

- This is a feasibility problem of finding a point in a polyhedral set C:

$$C = \{x \in \mathbb{R}^n \mid e_N' x = v_N, \quad e_S' x \geq v_S \text{ for } S \subset N\}$$

- Dynamic allocation is of interest where the players "negotiate" without a "central" entity
  - J.C. Cesco 1998 has proposed "transfer scheme" where coalitions are updating
  - E. Lehrer 2002 has considered "gradient- based" scheme where a randomly selected player updates at each time

- Both consider "repeated" static game (N, v) - optimization aspect obscured

Bargaining: Optimization perspective

- Solving feasibility problem distributedly among the players
- Define a **bounding set** X_i **of player** i:

$$X_i = \{x \in \mathbb{R}^n \mid e_N' x = v_N, \quad e_S' x \geq v_S \text{ for } S \subset N \text{ with } i \in S\}$$

- Note that

$$C = \{x \in \mathbb{R}^n \mid e_N' x = v_N, \quad e_S' x \geq v_S \text{ for } S \subset N\} = \cap_{i=1}^n X_i$$

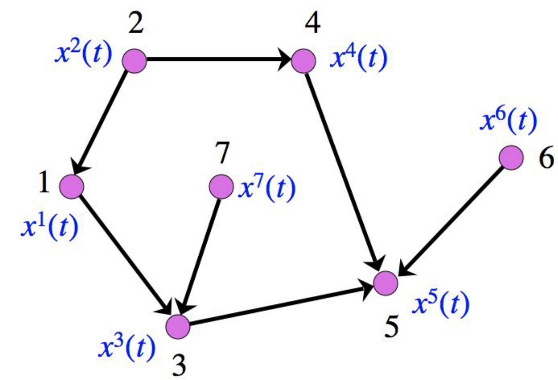
  - Possible formulation

$$\text{minimize} \quad \sum_{i=1}^n \text{dist}^2(x, X_i) = \sum_{i=1}^n \|x - \Pi_{X_i}[x]\|^2,$$

  where $\text{dist}(x, X)$ is the Euclidean distance from a point x to the set X and $\Pi_X[x]$ is the projection of x on X.
  - Iterative gradient method will work with
    - Incremental (cyclic) update or random player update
  - Distributed gradient method over a network will also work

TU Game over a Dynamic Network

- Players are viewed as nodes in a graph $(N, E(t))$
- Player j is a neighbor of i at time t if $(j, i) \in E(t)$
- $\mathcal{N}_i(t)$ is the set of neighbors of i at time t
- Allocation of player i at time t is $x^i(t)$
- Player i can see allocations $x^j(t)$ of his neighbors



Distributed over network bargaining where every player i updates using its bounding set X_i and allocations $x^j(t)$ of his neighbors $j \in \mathcal{N}_i(t)$:

$$w^i(t+1) = \underbrace{\sum_{j \in \mathcal{N}_i(t)} a_{ij}(t) x^j(t)}_{\text{alignment of allocations with neigbors}} \qquad a_{ij}(t) \geq 0, \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t) = 1$$

$$x^i(t+1) = \underbrace{\Pi_{X_i}[w^i(t+1)]}_{\text{gradient step to minimize } \mathrm{dist}^2(x, X_i)}$$

Convergence of such scheme will follow from a more general optimization setting*

*A. Nedić, A. Ozdaglar, P.A. Parrilo *Constrained consensus and optimization in multi-agent networks.* IEEE Trans. on Automatic Control, 55(4):922–938, 2010.
A. Nedić, J. Liu "On Convergence Rate of Weighted-Averaging Dynamics for Consensus Problems," under review, 2014

Bargaining: What if the players are not honest? What if the characteristic functions are random?

In some applications (supply chain, network controlled flows)

- The players may have an incentive to provide false information about v_i's trying to increase their own allocation values x_i
- The characteristic function v may depend on some random demand for a service of the players at any given time

This behavior leads us to consider **dynamic TU game** $(N, \{v(t)\})$, specified by

- Set $N = \{1, 2, \ldots, n\}$ of players
- Characteristic function $v(t)$ defining the instantaneous game $(N, v(t))$ at time t

In order to accommodate both situations, we assume that $v(t)$ **is random** and investigate

- **Robust game** - when uncertainty in $v(t)$ is unknown but bounded (in a way)
- **Averaging game** - under some ergodicity assumption on $\{v(t)\}$

Robust Game

Interested in a **bargaining process for dynamic TU game** $(N, \{v(t)\})$ **over a network**.

Assumption 1 *There exists* $v^{\max} \in \mathbb{R}^m$ *such that for all* $t \geq 0$*:*

$$v_S(t) \leq v_S^{\max} \quad \textit{for all } S \subset N, \qquad v_N(t) = v_N^{\max}.$$

The **robust TU game is the game** $(N, v^{\max})$.

Assumption 2 *The core* $C(v^{\max})$ *of the robust game is nonempty, i.e,*

$$C(v^{\max}) = \{x \in \mathbb{R}^n \mid e_N' x = v_N^{\max}, \;\; e_S' x \geq v_S^{\max} \textit{ for } S \subset N\} \neq \emptyset$$

- Instantaneous game $(N, v(t))$: player's bargaining involves time-varying bounding sets

$$X_i(t) = \{x \in \mathbb{R}^n \mid e_N' x = v_N^{\max}, \;\; e_S' x \geq v_S(t) \text{ for } S \subset N \text{ with } i \in S\}$$

- Bargaining protocol:

$$w^i(t+1) = \underbrace{\sum_{j \in \mathcal{N}_i(t)} a_{ij}(t) x^j(t)}_{\textbf{alignment of allocations with neigbors}} \qquad a_{ij}(t) \geq 0, \; \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t) = 1$$

$$x^i(t+1) = \underbrace{\Pi_{X_i(t)}[w^i(t+1)]}_{\textbf{gradient step to minimize } \mathrm{dist}^2(x, X_i(t))} \qquad \text{with arbitrary initial } x^i(0) \in \mathbb{R}^n$$

- **It is well defined**. Does it converge? If it does - where are the limit points?

Impact of the Network Connectivity Graphs

Assumption 3 *Assume that the graph* $(N, E(t))$ *is strongly connected*[†]. *Also, assume that* $a_{ij}(t) \geq 0$ *and* $\sum_{j \in N_i(t)} a_{ij}(t) = 1$ *for all* i *and* t. *In addition, there exists an* $\alpha > 0$ *such that* $a_{ii}(t) \geq \alpha$ *for all* t *and* $a_{ij}(t) \geq \alpha$ *whenever* $a_{ij}(t) > 0$.

- If only averaging

$$w^i(t+1) = \underbrace{\sum_{j \in \mathcal{N}_i(t)} a_{ij}(t) w^j(t)}_{\text{alignment of allocations with neigbors}} \qquad a_{ij}(t) \geq 0, \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t) = 1$$

- Semi-linear dynamics

$$w(t+1) = A(t)w(t) \qquad A(t) = [a(t)]_{ij} \text{ with 0-entries when } (j,i) \notin \mathcal{N}_i(t).$$

- Under Assumption 3, such dynamic will converge with geometric rate

- The limit point w^* is of the form $w_1^* = \cdots = w_n^*$

[†]Not critical. Strong connectivity over a period of time will work.

Bargaining Dynamic

$$w^i(t+1) = \underbrace{\sum_{j\in\mathcal{N}_i(t)} a_{ij}(t)x^j(t)}_{\text{alignment of allocations with neighbors}} \qquad a_{ij}(t) \geq 0, \sum_{j\in\mathcal{N}_i(t)} a_{ij}(t) = 1$$

$$x^i(t+1) = \underbrace{\Pi_{X_i(t)}[w^i(t+1)]}_{\text{gradient step to minimize } \mathrm{dist}^2(x, X_i(t))}$$

- Isolate the "linear" part

$$x^i(t+1) = w^i(t+1) + \underbrace{\Pi_{X_i(t)}[w^i(t+1)] - w^i(t+1)}_{\text{nonlinear error: } e^i(t)}$$

- Define $A(t) = [a(t)]_{ij}$ with 0-entries when $(j,i) \notin \mathcal{N}_i(t) = 0$

- Write it as "perturbed" semi-linear dynamic

$$x(t+1) = A(t)x(t) + e(t)$$

- Under Assum.4, the convergence of such dynamics will depend on the behavior of $e(t)$

Convergence to the Core of the Robust Game

Let Assumptions 1–3 hold. Also, assume that

$$\mathrm{Prob}\,\{v(t) = v^{\max} \text{ infinitely often}\} = 1.$$

Then, the bargaining protocol converges to a (random) allocation in the core $C(v^{\max})$ with probability 1.

- Consider $y(t) = \frac{1}{n}\sum_{i=1}^{m} x^i(t)$
- $\|y(t) - x^i(t)\| \to 0$ for all i (non-emptiness of the core $C(v^{\max})$)
- $\{\|y(t) - x\|\}$ convergent w.p. 1 for any $x \in C(v^{\max})$
- Critical observation: bounding sets (and the core) of instantaneous game have the same "normals"

  $$X_i(t) = \{x \in \mathbb{R}^n \mid e_N' x = v_N^{\max}, \quad e_S' x \geq v_S(t) \text{ for } S \subset N \text{ with } i \in S\}$$

  As a consequence (by Hoffman's bound)

  $$\mathrm{dist}^2(y(t+1), C(v(t)) \leq \mu \sum_{i=1}^{n} \mathrm{dist}^2(y(t+1), X_i(t))$$

Dynamic Average Game

Consider dynamic TU game $(N, \{v(t)\})$

- Define
$$\bar{v}(t) = \frac{1}{t+1} \sum_{k=0}^{t} v(k) \qquad \text{for all} \quad t \geq 0$$

- Average instantaneous game $(N, \bar{v}(t))$

- Bounding sets
$$\bar{X}_i(t) = \{x \in \mathbb{R}^n \mid e_N' x = \bar{v}_N(t), \quad e_S' x \geq \bar{v}_S(t) \text{ for } S \subset N \text{ with } i \in S\}$$

- Bargaining protocol:
$$w^i(t+1) = \underbrace{\sum_{j \in \mathcal{N}_i(t)} a_{ij}(t) x^j(t)}_{\text{alignment of allocations with neighbors}} \qquad a_{ij}(t) \geq 0, \ \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t) = 1$$
$$x^i(t+1) = \underbrace{\Pi_{\bar{X}_i(t)}[w^i(t+1)]}_{\text{gradient step to minimize } \mathrm{dist}^2(x, \bar{X}_i(t))}$$

Average TU Game

Assumption 4 *With probability 1, we have*

$$\lim_{t\to\infty} \bar{v}(t) = v^{\rm mean}, \qquad v_N(t) = v_N^{\rm mean} \quad \textit{for all } t \geq 0$$

- Average game $(N, v^{\rm mean})$ with the core $C(v^{\rm mean})$:

$$C(v^{\rm mean}) = \{x \in \mathbb{R}^n \mid e_N' x = v_N^{\rm mean}, \;\; e_S' x \geq v_S^{\rm mean} \text{ for } S \subset N\} \neq \emptyset$$

Let Assumptions 3 and 4 hold. Assume also that $\dim C(v^{\rm mean}) = n - 1$. Then, the bargaining process converges to a (random) allocation in the core $C(v^{\rm mean})$ of the average game with probability 1.

Proof Sketch

- Consider $y(t) = \frac{1}{n}\sum_{i=1}^{m} x^i(t)$

- Assumption that $\dim C(v^{\rm mean}) = n-1$ implies: for every $z \in {\rm relint} C(v^{\rm mean})$ there is t_z large enough so that $z \in C(\bar v(t))$ for all $t \ge t_z$ with probability 1

- The preceding helps establish
  - $\{\|y(t) - z\|\}$ convergent w.p. 1 for any $z \in {\rm relint} C(v^{\rm mean})$
  - $\|y(t) - x^i(t)\| \to 0$ for all i $\quad\Longrightarrow\quad$ ${\rm dist}(y(t+1), \bar X_i(t)) \to 0$

- Bounding sets of the instantaneous average game have the same "normals"

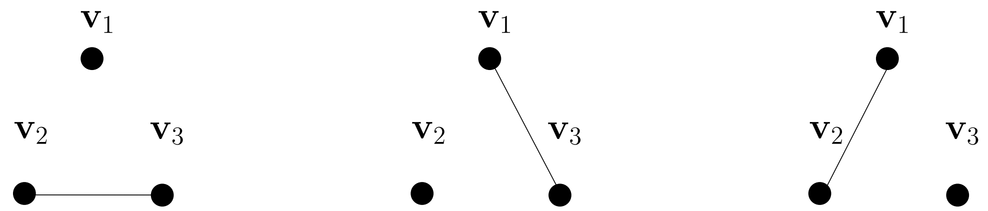
$$\bar X_i(t) = \{x \in \mathbb{R}^n \mid e_N' x = v_N^{\rm mean}, \quad e_S' x \ge \bar v_S(t) \text{ for } S \subset N \text{ with } i \in S\}$$

  As a consequence, for some $L_i > 0$ w.p. 1 for any x, any i and all t

$${\rm dist}(x, \bar X_i) \le {\rm dist}(x, \bar X_i(t)) + L_i \|\bar v(t) - v^{\rm mean}\|$$

- This yields ${\rm dist}(y(t+1), \bar X_i) \to 0$ for all i w.p. 1

Numerical Examples



(a) (b) (c)

$$A(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \qquad A(1) = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \qquad A(2) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- **Robust game**: flip a fair coin; if head then choose $v(t)$ with uniform distribution, otherwise choose $v^{\max}$
  - The core of the robust game $C(v^{\max}) = \{(7, 3, 0, 0, 0, 0, 10)\}$
- **Average game**: we always use uniform distribution over the given intervals

	$v_{\{1\}}$	$v_{\{2\}}$	$v_{\{3\}}$	$v_{\{1,2\}}$	$v_{\{1,3\}}$	$v_{\{2,3\}}$	$v_{\{1,2,3\}}$
Robust game	$[4, 7]$	$[0, 3]$	0	0	0	0	10
Average game	$[4, 9]$	$[0, 5]$	0	0	0	0	10

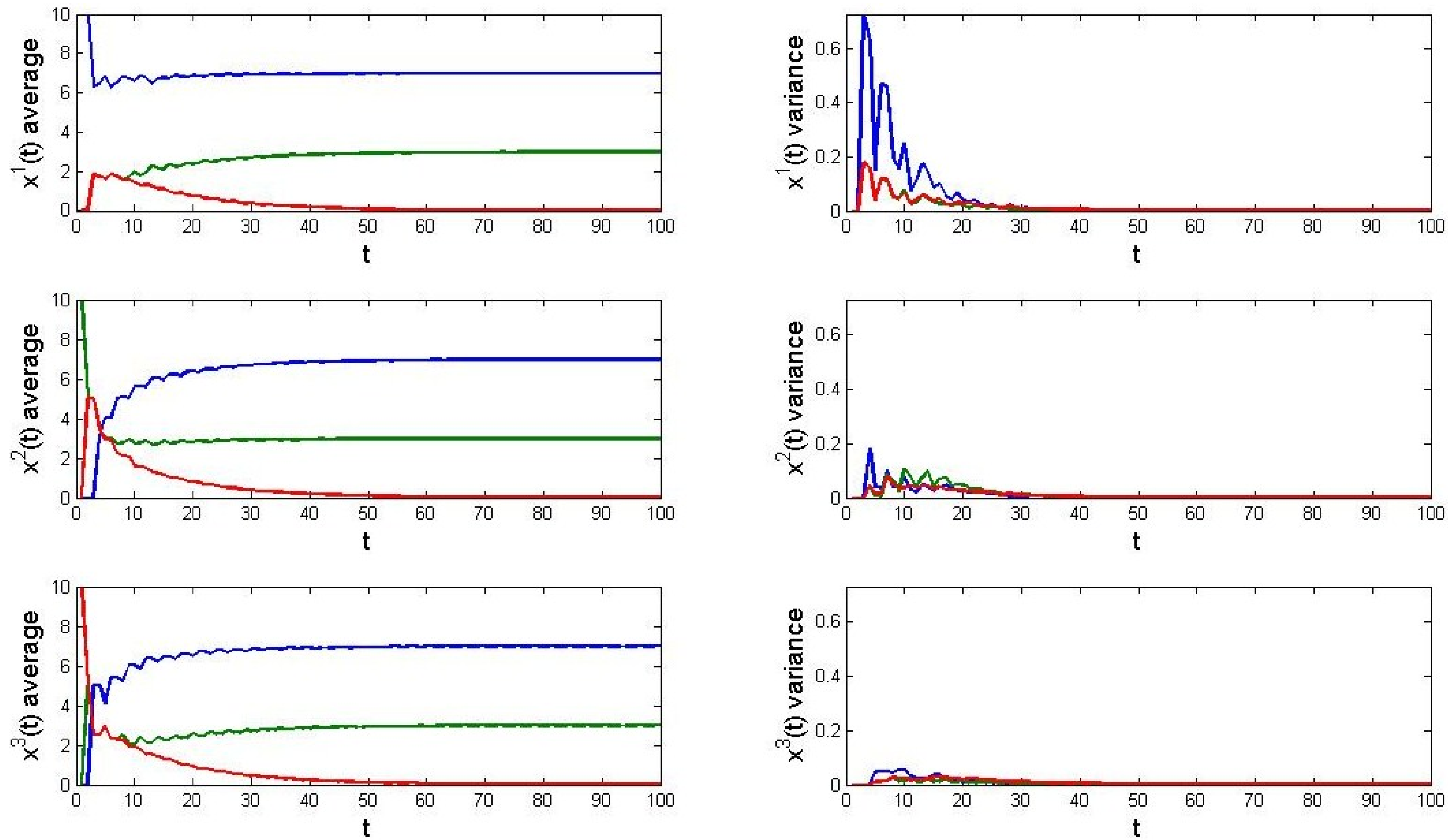


Figure 1: Robust Game Results blue - player 1, green - player 2, Initial Allocation: selfish

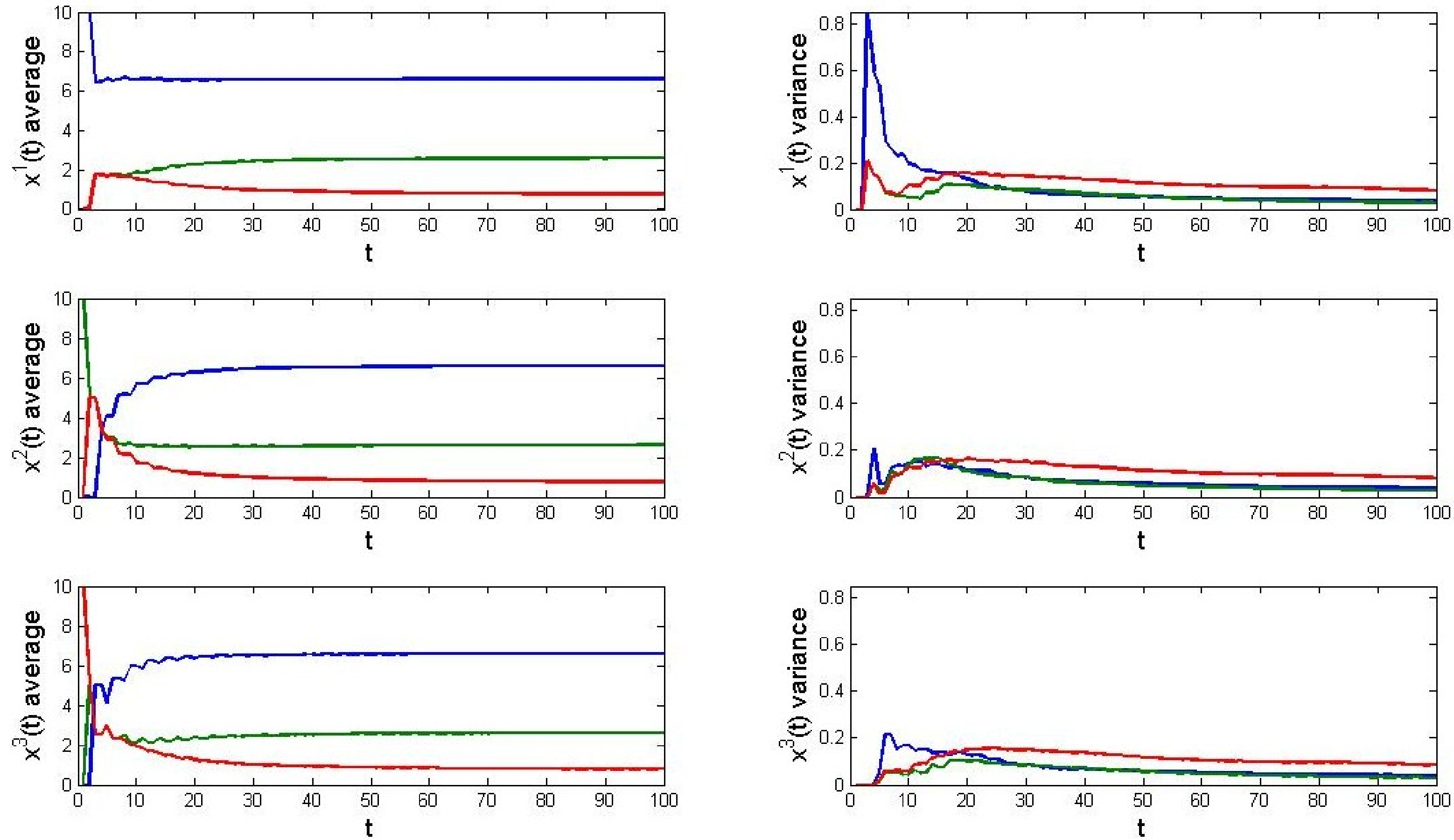


Figure 2: Average Game Results blue - player 1, green - player 2, Initial Allocation: selfish

Conclusion

- Considered dynamic TU games over networks: dynamic in the game and in the player's network
- Main assumption: grand coalition is stable for some well defined "limiting game"
- Bargaining protocols converge to an allocation in the core of the limiting game

Future Directions

- Other dynamic games such as zero-sum games, potential games etc.
- Framework and algorithms needed