Lecture 8

Strong Duality Results

September 22, 2008
Outline

• Slater Condition and its Variations

• Convex Objective with Linear Inequality Constraints

• Quadratic Objective over Quadratic Constraints

• Representation Issue

• Multiple Dual Choices
General Convex Problem

Primal Problem

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g_j(x) \leq 0, \quad j = 1, \ldots, m \\
& \quad a_i^T x = b_i, \quad i = 1, \ldots, r \\
& \quad x \in X
\end{align*}
\]

Assumption used throughout the lecture

Assumption 1.

- The objective \( f \) and all \( g_j \) are convex
- The set \( X \subseteq \mathbb{R}^n \) is nonempty and convex
- The optimal value \( f^* \) is finite
Slater Condition for Inequality Constrained Problem

Convex Primal Problem

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g_j(x) \leq 0, \quad j = 1, \ldots, m \\
& \quad x \in X 
\end{align*}
\]

Slater Condition There exists a vector \( \bar{x} \in X \cap \text{dom} f \) such that

\[
g_j(\bar{x}) < 0 \quad \text{for all } j = 1, \ldots, m.
\]

Observation: The Slater condition is a condition on the constraint set only when \( \text{dom} f = \mathbb{R}^n \)

Theorem Let Assumption 1 and the Slater condition hold. Then:

- There is no duality gap, i.e., \( q^* = f^* \)
- The set of dual optimal solutions is nonempty and bounded
Proof

Consider the set $V \subset \mathbb{R}^m \times \mathbb{R}$ given by

$$V = \{(u, w) \mid g(x) \leq u, \; f(x) \leq w, \; x \in X\}$$

The Slater condition is illustrated in the figures below.

The set $V$ is convex by the convexity of $f$, $g'_j$s, and $X$. 
**Proof continues**

The vector \((0, f^*)\) is not in the interior of the set \(V\). Suppose it is, i.e., \((0, f^*) \in \text{int}V\). Then, there exists an \(\epsilon > 0\) such that \((0, f^* - \epsilon) \in V\) contradicting the optimality of \(f^*\).

Thus, either \((0, f^*) \in \text{bd}V\) or \((0, f^*) \notin V\). By the Supporting Hyperplane Theorem, there exists a hyperplane passing through \((0, f^*)\) and supporting the set \(V\): there exists \((\mu, \mu_0) \in \mathbb{R}^m \times \mathbb{R}\) with \((\mu, \mu_0) \neq 0\) such that

\[
\mu^T u + \mu_0 w \geq \mu_0 f^* \quad \text{for all} \ (u, w) \in V
\]

This relation implies that \(\mu \succeq 0\) and \(\mu_0 \geq 0\).

Suppose that \(\mu_0 = 0\). Then, \(\mu \neq 0\) and relation (1) reduces to

\[
\inf_{(u, v) \in V} \mu^T u = 0
\]

On the other hand, by the definition of the set \(V\), since \(\mu \succeq 0\) and \(\mu \neq 0\), we have

\[
\inf_{(u, v) \in V} \mu^T u = \inf_{x \in X} \mu^T g(x) \leq \mu^T g(\bar{x}) < 0 \quad \text{- a contradiction}
\]
Hence, $\mu_0 > 0$ (the supporting hyperplane is nonvertical), and by dividing with $\mu_0$ in Eq. (1), we obtain

$$\inf_{(u,v) \in V} \{\tilde{\mu}^T u + w\} \geq f^*$$

with $\tilde{\mu} \geq 0$. Therefore,

$$q(\tilde{\mu}) = \inf_{x \in X} \{f(x) + \tilde{\mu}^T g(x)\} \geq f^* \quad \text{with } \tilde{\mu} \geq 0$$

implying that $q^* \geq f^*$. By the weak duality [$q^* \leq f^*$], it follows that $q^* = f^*$ and $\tilde{\mu}$ is a dual optimal solution.
We now show that the set of dual optimal solutions is bounded. For any dual optimal $\tilde{\mu} \succeq 0$, we have

$$q^* = q(\tilde{\mu}) = \inf_{x \in X} \left\{ f(x) + \tilde{\mu}^T g(x) \right\}$$

$$\leq f(\bar{x}) + \tilde{\mu}^T g(\bar{x})$$

$$\leq f(\bar{x}) + \max_{1 \leq j \leq m} \{ g_j(\bar{x}) \} \sum_{j=1}^{m} \tilde{\mu}_j$$

Therefore,

$$\min_{1 \leq j \leq m} \{ -g_j(\bar{x}) \} \sum_{j=1}^{m} \tilde{\mu}_j \leq f(\bar{x}) - q^*$$

implying that

$$\|\tilde{\mu}\| \leq \sum_{j=1}^{m} \tilde{\mu}_j \leq \frac{f(\bar{x}) - q^*}{\min_{1 \leq j \leq m} \{ -g_j(\bar{x}) \}}$$
Example

\( l_\infty \)-Norm Minimization

\[
\text{minimize } \|Ax - b\|_\infty
\]

Equivalent to

\[
\begin{align*}
\text{minimize } & \quad t \\
\text{subject to } & \quad a_j^T x - b_j - t \leq 0, \ j = 1, \ldots, m \\
& \quad b_j - a_j^T x - t \leq 0, \ j = 1, \ldots, m \\
( & \quad (x, t) \in \mathbb{R}^n \times \mathbb{R})
\end{align*}
\]

The vector \((\bar{x}, \bar{t})\) given by

\[
\bar{x} = 0 \quad \text{and} \quad \bar{t} = \epsilon + \max_{1 \leq j \leq m} |b_j| \quad \text{for some } \epsilon > 0
\]

satisfies the Slater condition

We refer to a vector satisfying the Slater condition as a \textit{Slater vector}
**Convex Objective Over Linear Constraints**

**Def.** A vector \( x_0 \) is in the relative interior of \( X \) when there exists a ball \( B_r(x_0) = \{ x \in \mathbb{R}^n \mid \| x - x_0 \| \leq r \} \) such that
\[
B_r(x_0) \cap \text{aff}(X) \subseteq X
\]

The relative interior of a convex set \( X \) is the set of points that are interior with respect to the affine hull of \( X \).

**Primal Problem with Linear Inequality Constraints**

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad Ax \leq b, \\
& \quad x \in X
\end{align*}
\]

**Theorem** Let Assumption 1 hold. Assume that there exists a vector \( \tilde{x} \) such that
\[
A\tilde{x} \leq b \quad \text{and} \quad \tilde{x} \in \text{relint}(X) \cap \text{relint}(\text{dom } f)
\]

Then
- There is no duality gap \( (q^* = f^*) \) and
- A dual optimal solution exists
Implications

• When \( \text{dom} f = \mathbb{R}^n \), the relative interior condition has to do only with the constraint set.

• The “relative interior” condition is always satisfied
  
  (a) When \( \text{dom} f = \mathbb{R}^n \) and \( X = \mathbb{R}^n \):
  convex function over \( \mathbb{R}^n \) and linear constraints
  
  (b) LP’s: \( X = \mathbb{R}^n \) and \( f(x) = c^T x \).

• In these cases, by the preceding theorem:
  
  • Strong duality holds and optimal dual solution exists
Examples

Examples not satisfying the “relative interior” condition

\[
\begin{align*}
\text{minimize} & \quad e^{-\sqrt{x_1x_2}} \\
\text{subject to} & \quad x_1 \leq 0, \quad x \in \mathbb{R}^2
\end{align*}
\]

Here, \( X = \mathbb{R}^2 \) and the feasible set \( C \) is \( C = \{ x \mid x_1 = 0, \ x_2 \in \mathbb{R} \} \). The domain of \( f \) is the set \( \{ x \mid x_1 \geq 0, \ x_2 \geq 0 \} \), thus,
\[
\text{relint}(X) \cap \text{relint}(\text{dom}f) = \{ x \mid x_1 > 0, \ x_2 > 0 \}.
\]

However, none of the feasible vectors lies in \( \text{relint}(X \cap \text{dom}f) \). Hence, the condition fails. Furthermore, there is a gap - Homework.

Consider the same constraint set with an objective \( f(x) = -\sqrt{x_1} \). The condition is not satisfied. In this case, there is no gap, but a dual optimal solution does not exist.
**Slater for a General Convex Problem**

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g_j(x) \leq 0, \quad j = 1, \ldots, m \\
& \quad Ax = b, \quad Dx \preceq d \\
& \quad x \in X
\end{align*}
\]

**Theorem** Let \( f^* \) be finite, and assume that:

- There is a feasible vector satisfying the Slater condition, i.e., there is \( \bar{x} \in \mathbb{R}^n \) such that
  
  \[ g(\bar{x}) \prec 0, \quad A\bar{x} = b, \quad D\bar{x} \preceq d, \quad \bar{x} \in X \cap \text{dom} f \]

- There is a vector satisfying the linear constraints and belongs to the relative interior of \( X \), i.e., there is \( \tilde{x} \in \mathbb{R}^n \) such that
  
  \[ A\tilde{x} = b, \quad D\tilde{x} \preceq d, \quad \tilde{x} \in \text{relint}(X) \cap \text{relint}(\text{dom} f) \]

Then:

- There is no duality gap, i.e., \( q^* = f^* \)
- The set of dual optimal solutions is nonempty

**Note:** Conditions only on the constraint set

Theorem says nothing about the existence of primal optimal solutions
Quadratic Objective over Quadratic Constraint

minimize \( x^T Q_0 x + a_0^T x + b_0 \)
subject to \( x^T Q_j x + a_j^T x + b_j \leq 0, \quad j = 1, \ldots, m \)
\( x \in \mathbb{R}^n \)

**Theorem** Let each \( Q_i \) be symmetric positive semidefinite matrix. Let \( f^* \) be finite. Then, there is no duality gap \([q^* = f^*]\) and the primal optimal set is nonempty.

**Note:**
- The theorem says nothing about the existence of dual optimal solutions
- Opposed to the Slater and the relative interior condition, which say nothing about the existence of the primal optimal solutions

**Example**

minimize \( \|x\| \)
subject to \( Ax = b \)
\( x \in \mathbb{R}^n \)
Representation Issue

minimize $-x_2$
subject to $\|x\| \leq x_1$

$x \in X, X = \{(x_1, x_2) | x_2 \geq 0\}$

- The relaxation of the inequality constraint results in a dual problem, for which the dual value $q(\mu)$ is $-\infty$ for any $\mu \geq 0$ (verify yourself).
  Thus, $q^* = -\infty$, while $f^* = 0$.

However, a closer look into the constraint set reveals

$$ C = \{(x_1, x_2) | x_1 \geq 0, x_2 = 0\} $$

Thus the problem is equivalent to

minimize $-x_2$
subject to $x_1 \geq 0, x_2 = 0$

$x \in \mathbb{R}^2$

- There is no gap for this problem!!! Why?

The duality gap issue is closely related to the “representation” of the constraints [the model for the constraints]
Multiple Choices for a Dual Problem

\[
\begin{align*}
\text{minimize} \quad & f(x) \\
\text{subject to} \quad & g_j(x) \leq 0, \quad j = 1, \ldots, m \\
& Ax = b, \quad Dx \preceq d \\
& x \in X
\end{align*}
\]

- There are multiple choices for the set of “inequalities” to be relaxed
- How to choose the “right one”? There is no general rule
  - Keep box constraints, “sign” constraints if any
  - Keep constraints for which the dual function can be easily evaluated
- Will for all of them “no duality-gap” relation hold?
- What do we know about duality gaps when “multiple dual choices” exist?
Relax-All Rule

minimize $f(x)$
subject to $g_j(x) \leq 0$, $j = 1, \ldots, m$

$x \in \mathbb{R}^n$

A linear equality is represented by two linear inequalities.

**Theorem** Let $f^*$ be finite. Consider a dual corresponding to the relaxation of all the constraints, and assume that there is no duality gap. Then, there is no duality gap when partially relaxing the constraints.