# Lecture 8

## Strong Duality Results

September 22, 2008

# Outline

- Slater Condition and its Variations
- Convex Objective with Linear Inequality Constraints
- Quadratic Objective over Quadratic Constraints
- Representation Issue
- Multiple Dual Choices

# General Convex Problem

## Primal Problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_j(x) \leq 0, \quad j = 1, \ldots, m \\ & a_i^T x = b_i, \quad i = 1, \ldots, r \\ & x \in X \end{array}$$

## Assumption used throughout the lecture

**Assumption 1.**

- The objective $f$ and all $g_j$ are convex
- The set $X \subseteq \mathbb{R}^n$ is nonempty and convex
- The optimal value $f^*$ is finite

# Slater Condition for Inequality Constrained Problem

## Convex Primal Problem

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_j(x) \le 0, \quad j = 1, \dots, m \\ & x \in X \end{aligned}$$

**Slater Condition** There exists a vector $\bar{x} \in X \cap \mathrm{dom} f$ such that

$$g_j(\bar{x}) < 0 \quad \text{for all } j = 1, \dots, m.$$

*Observation:* The Slater condition is a condition on the constraint set only when $\mathrm{dom} f = \mathbb{R}^n$

**Theorem** Let Assumption 1 and the Slater condition hold. Then:

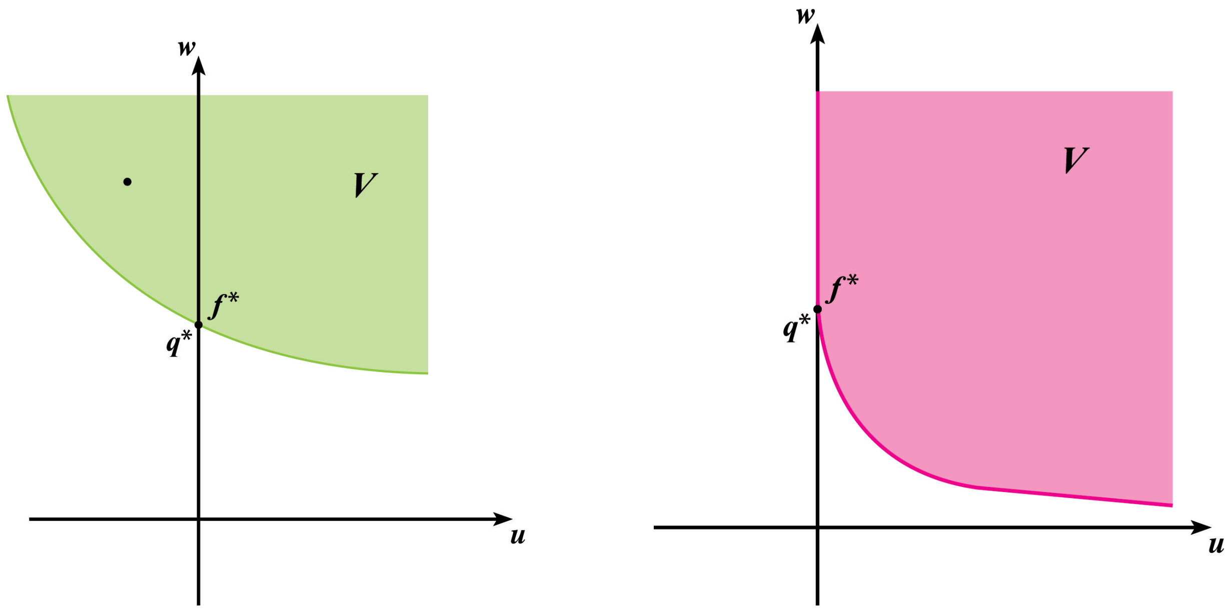
- There is no duality gap, i.e., $q^* = f^*$
- The set of dual optimal solutions is nonempty and bounded

# Proof

Consider the set $V \subset \mathbb{R}^m \times \mathbb{R}$ given by

$$V = \{(u, w) \mid g(x) \preceq u,\ f(x) \leq w,\ x \in X\}$$

The Slater condition is illustrated in the figures below.




The set $V$ is convex by the convexity of $f$, $g_j' s$, and $X$.

# Proof continues

The vector $(0, f^*)$ is not in the interior of the set $V$. Suppose it is, i.e., $(0, f^*) \in \mathrm{int}V$. Then, there exists an $\epsilon > 0$ such that $(0, f^* - \epsilon) \in V$ contradicting the optimality of $f^*$.

Thus, either $(0, f^*) \in \mathrm{bd}V$ or $(0, f^*) \notin V$. By the Supporting Hyperplane Theorem, there exists a hyperplane passing through $(0, f^*)$ and supporting the set $V$: there exists $(\mu, \mu_0) \in \mathbb{R}^m \times \mathbb{R}$ with $(\mu, \mu_0) \neq 0$ such that

$$\mu^T u + \mu_0 w \geq \mu_0 f^* \qquad \text{for all } (u, w) \in V \tag{1}$$

This relation implies that $\mu \succeq 0$ and $\mu_0 \geq 0$.

Suppose that $\mu_0 = 0$. Then, $\mu \neq 0$ and relation (1) reduces to

$$\inf_{(u,v) \in V} \mu^T u = 0$$

On the other hand, by the definition of the set $V$, since $\mu \succeq 0$ and $\mu \neq 0$, we have

$$\inf_{(u,v) \in V} \mu^T u = \inf_{x \in X} \mu^T g(x) \leq \mu^T g(\bar{x}) < 0 \qquad \text{- a contradiction}$$

Hence, $\mu_0 > 0$ (the supporting hyperplane is nonvertical), and by dividing with $\mu_0$ in Eq. (1), we obtain

$$\inf_{(u,v)\in V} \{\tilde{\mu}^T u + w\} \geq f^*$$

with $\tilde{\mu} \succeq 0$. Therefore,

$$q(\tilde{\mu}) = \inf_{x\in X} \left\{f(x) + \tilde{\mu}^T g(x)\right\} \geq f^* \qquad \text{with } \tilde{\mu} \succeq 0$$

implying that $q^* \geq f^*$. By the weak duality [$q^* \leq f^*$], it follows that $q^* = f^*$ and $\tilde{\mu}$ is a dual optimal solution.

We now show that the set of dual optimal solutions is bounded. For any dual optimal $\tilde{\mu} \succeq 0$, we have

$$
\begin{aligned}
q^* = q(\tilde{\mu}) &= \inf_{x \in X} \left\{ f(x) + \tilde{\mu}^T g(x) \right\} \\
&\le f(\bar{x}) + \tilde{\mu}^T g(\bar{x}) \\
&\le f(\bar{x}) + \max_{1 \le j \le m} \{ g_j(\bar{x}) \} \sum_{j=1}^{m} \tilde{\mu}_j
\end{aligned}
$$

Therefore, $\min_{1 \le j \le m} \{ -g_j(\bar{x}) \} \sum_{j=1}^{m} \tilde{\mu}_j \le f(\bar{x}) - q^*$ implying that

$$
\|\tilde{\mu}\| \le \sum_{j=1}^{m} \tilde{\mu}_j \le \frac{f(\bar{x}) - q^*}{\min_{1 \le j \le m} \{ -g_j(\bar{x}) \}}
$$

# Example

**$l_\infty$-Norm Minimization**

$$\text{minimize} \;\; \|Ax - b\|_\infty$$

Equivalent to

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a_j^T x - b_j - t \le 0, \;\; j = 1, \ldots, m \\ & b_j - a_j^T x - t \le 0, \;\; j = 1, \ldots, m \\ & (x, t) \in \mathbb{R}^n \times \mathbb{R} \end{array}$$

The vector $(\bar{x}, \bar{t}\,)$ given by

$$\bar{x} = 0 \quad \text{and} \quad \bar{t} = \epsilon + \max_{1 \le j \le m} |b_j| \quad \text{for some } \epsilon > 0$$

satisfies the Slater condition

We refer to a vector satisfying the Slater condition as a *Slater vector*

# Convex Objective Over Linear Constraints

**Def.** A vector $x_0$ is in the relative interior of $X$ when there exists a ball $B_r(x_0) = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$ such that

$$B_r(x_0) \cap \mathrm{aff}(X) \subseteq X$$

The relative interior of a convex set $X$ is the set of points that are interior with respect to the affine hull of $X$.

## Primal Problem with Linear Inequality Constraints

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & Ax \leq b, \\ & x \in X \end{aligned}$$

**Theorem** Let Assumption 1 hold. Assume that there exists a vector $\tilde{x}$ such that

$$A\tilde{x} \leq b \qquad \text{and} \qquad \tilde{x} \in \mathrm{relint}(X) \cap \mathrm{relint}(\mathrm{dom} f)$$

Then

- There is no duality gap ($q^* = f^*$) and
- A dual optimal solution exists

# Implications

- When $\text{dom}f = \mathbb{R}^n$, the relative interior condition has to do only with the constraint set.

- The "relative interior" condition is always satisfied

  (a) When $\text{dom}f = \mathbb{R}^n$ and $X = \mathbb{R}^n$:
  convex function over $\mathbb{R}^n$ and linear constraints

  (b) LP's: $X = \mathbb{R}^n$ and $f(x) = c^T x$.

- In these cases, by the preceding theorem:
  - Strong duality holds and optimal dual solution exists

# Examples

*Examples not satisfying the "relative interior" condition*

$$\begin{aligned}&\text{minimize} \quad e^{-\sqrt{x_1 x_2}}\\&\text{subject to } x_1 \leq 0, \quad x \in \mathbb{R}^2\end{aligned}$$

Here, $X = \mathbb{R}^2$ and the feasible set $C$ is $C = \{x \mid x_1 = 0,\ x_2 \in \mathbb{R}\}$. The domain of $f$ is the set $\{x \mid x_1 \geq 0,\ x_2 \geq 0\}$, thus,
$\mathrm{relint}(X) \cap \mathrm{relint}(\mathrm{dom} f) = \{x \mid x_1 > 0,\ x_2 > 0\}$.
However, none of the feasible vectors lies in $\mathrm{relint}(X \cap \mathrm{dom} f)$. Hence, the condition fails. Furthermore, **there is a gap** - Homework.

Consider the same constraint set with an objective $f(x) = -\sqrt{x_1}$. The condition is not satisfied. In this case, there is no gap, but a dual optimal solution does not exist.

# Slater for a General Convex Problem

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_j(x) \le 0, \quad j = 1, \ldots, m \\ & Ax = b, \; Dx \preceq d \\ & x \in X \end{aligned}$$

**Theorem** Let $f^*$ be finite, and assume that:

- There is a feasible vector satisfying the Slater condition, i.e., there is $\bar{x} \in \mathbb{R}^n$ such that
$$g(\bar{x}) \prec 0, \quad A\bar{x} = b, \quad D\bar{x} \preceq d, \quad \bar{x} \in X \cap \mathrm{dom} f$$
- There is a vector satisfying the linear constraints and belongs to the relative interior of $X$, i.e., there is $\tilde{x} \in \mathbb{R}^n$ such that
$$A\tilde{x} = b, \quad D\tilde{x} \preceq d, \quad \tilde{x} \in \mathrm{relint}(X) \cap \mathrm{relint}(\mathrm{dom} f)$$

Then:

- There is no duality gap, i.e., $q^* = f^*$
- The set of dual optimal solutions is nonempty

*Note:* Conditions only on the constraint set

Theorem says nothing about the existence of primal optimal solutions

# Quadratic Objective over Quadratic Constraint

$$\begin{array}{ll} \text{minimize} & x^T Q_0 x + a_0^T x + b_0 \\ \text{subject to} & x^T Q_j x + a_j^T x + b_j \leq 0, \quad j = 1, \ldots, m \\ & x \in \mathbb{R}^n \end{array}$$

**Theorem** Let each $Q_i$ be symmetric positive semidefinite matrix. Let $f^*$ be finite. Then, there is no duality gap $[q^* = f^*]$ and the primal optimal set is nonempty.

*Note:*

- The theorem says nothing about the existence of dual optimal solutions
- Opposed to the Slater and the relative interior condition, which say nothing about the existence of the primal optimal solutions

*Example*

$$\begin{array}{ll} \text{minimize} & \|x\| \\ \text{subject to} & Ax = b \\ & x \in \mathbb{R}^n \end{array}$$

# Representation Issue

$$\begin{array}{ll} \text{minimize} & -x_2 \\ \text{subject to} & \|x\| \le x_1 \\ & x \in X, \;\; X = \{(x_1, x_2) \mid x_2 \ge 0\} \end{array}$$

- The relaxation of the inequality constraint results in a dual problem, for which the dual value $q(\mu)$ is $-\infty$ for any $\mu \ge 0$ (verify yourself).
  Thus, $q^* = -\infty$, while $f^* = 0$.
  However, a closer look into the constraint set reveals
  $$C = \{(x_1, x_2) \mid x_1 \ge 0, \;\; x_2 = 0\}$$
  Thus the problem is equivalent to
  $$\begin{array}{ll} \text{minimize} & -x_2 \\ \text{subject to} & x_1 \ge 0, \;\; x_2 = 0 \\ & x \in \mathbb{R}^2 \end{array}$$
- There is no gap for this problem!!! Why?

**The duality gap issue is closely related to the "representation" of the constraints** [the model for the constraints]

# Multiple Choices for a Dual Problem

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_j(x) \le 0, \quad j = 1, \ldots, m \\ & Ax = b, \ Dx \preceq d \\ & x \in X \end{aligned}$$

- There are multiple choices for the set of "inequalities" to be relaxed
- How to choose the "right one"? There is no general rule
  - Keep box constraints, "sign" constraints if any
  - Keep constraints for which the dual function can be easily evaluated
- Will for all of them "no duality-gap" relation hold?
- What do we know about duality gaps when "multiple dual choices" exist?

# Relax-All Rule

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_j(x) \le 0, \;\; j = 1, \ldots, m \\ & x \in \mathbb{R}^n \end{array}$$

A linear equality is represented by two linear inequalities.

**Theorem** Let $f^*$ be finite. Consider a dual corresponding to the relaxation of all the constraints, and assume that there is no duality gap. Then, there is no duality gap when partially relaxing the constraints.