Lecture 6
Convex Problems

September 15, 2008

- Chapters 4.1-4.2 (except for 4.2.5) of Boyd and Vandenberghe's book

Introducing Slack Variables for Linear Inequalities

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax \preceq b \end{array}$$

with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, is equivalent to

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax + s = b \\ & s \succeq 0 \end{array}$$

- The minimization is over $x \in \mathbb{R}^n$ and $s \in \mathbb{R}^m$
- Convexity preserved

Convex Problem with Inequalities and Equalities

$$\begin{array}{ll}\text{minimize} & f(x)\\ \text{subject to} & g(x) \le 0, \; Ax = b\end{array}$$

- *Eliminating equality constraints*:
  For a particular solution x_0 of $Ax = b$, and a matrix D whose range is N_A, we have
  $$Ax = b \quad \Longleftrightarrow \quad x = Dz + x_0 \text{ for some } z$$
- The problem is equivalent to

$$\begin{array}{ll}\text{minimize} & f(Dz + x_0)\\ \text{subject to} & g(Dz + x_0) \le 0\end{array}$$

- The minimization is over z
- Convexity preserved

Introducing Equality Constraints

$$\begin{array}{ll} \text{minimize} & f(A_0 x + b_0) \\ \text{subject to} & g_j(A_j x + b_j) \leq 0, \quad j = 1, \ldots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize} & f(y_0) \\ \text{subject to} & g_j(y_j) \leq 0, \quad j = 1, \ldots, m \\ & y_j = A_j x + b_j, \quad j = 0, 1, \ldots, m \end{array}$$

- The minimization is over x and y_j, $j = 0, 1, \ldots, m$
- Convexity preserved

Epigraph Form

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \preceq 0, \;\; Ax = b \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize} & w \\ \text{subject to} & f(x) \le w \\ & g(x) \preceq 0, \;\; Ax = b \end{array}$$

- The minimization is over $x \in \mathbb{R}^n$ and $w \in \mathbb{R}$
- Is convexity preserved?

Polyhedral Objective Example

$$\begin{array}{ll} \text{minimize} & \max\{a_1^T x + b_1, \ldots, a_m^T x + b_m\} \\ \text{subject to} & B^T x \le d \end{array}$$

- Using the epigraph form of f, the problem is equivalent to an LP

$$\begin{array}{ll} \text{minimize} & w \\ \text{subject to} & a_j^T x + b_j \le w, \quad j = 1, \ldots, m \\ & B^T x \le d \end{array}$$

Minimizing over some Variables

$$\begin{array}{ll} \text{minimize} & f(x_1, x_2) \\ \text{subject to} & g_j(x_1) \leq 0, \quad j = 1, \ldots, \tilde{m} \\ & g_j(x_2) \leq 0, \quad j = \tilde{m}+1, \ldots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize} & F(x_2) \\ \text{subject to} & g_j(x_2) \leq 0, \quad j = \tilde{m}+1, \ldots, m \end{array}$$

where

$$F(x_2) = \inf_{x_1:\ g_1(x_1) \leq 0, \ldots, g_{\tilde{m}}(x_1) \leq 0} f(x_1, x_2)$$