Lecture 6
Convex Problems

September 15, 2008

- Chapters 4.1-4.2 (except for 4.2.5) of Boyd and Vandenberghe’s book
Introducing Slack Variables for Linear Inequalities

minimize \( f(x) \)
subject to \( Ax \leq b \)

with \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \), is equivalent to

minimize \( f(x) \)
subject to \( Ax + s = b \)
\( s \geq 0 \)

- The minimization is over \( x \in \mathbb{R}^n \) and \( s \in \mathbb{R}^m \)
- Convexity preserved
Convex Problem with Inequalities and Equalities

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g(x) \leq 0, \ Ax = b
\end{align*}
\]

- Eliminating equality constraints:
  For a particular solution \(x_0\) of \(Ax = b\), and a matrix \(D\) whose range is \(N_A\), we have
  \[
  Ax = b \iff x = Dz + x_0 \text{ for some } z
  \]
- The problem is equivalent to
  \[
  \begin{align*}
  \text{minimize} & \quad f(Dz + x_0) \\
  \text{subject to} & \quad g(Dz + x_0) \leq 0
  \end{align*}
  \]
- The minimization is over \(z\)
- Convexity preserved
Introducing Equality Constraints

\[
\begin{align*}
\text{minimize} & \quad f(A_0x + b_0) \\
\text{subject to} & \quad g_j(A_jx + b_j) \leq 0, \quad j = 1, \ldots, m
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\text{minimize} & \quad f(y_0) \\
\text{subject to} & \quad g_j(y_j) \leq 0, \quad j = 1, \ldots, m \\
& \quad y_j = A_jx + b_j, \quad j = 0, 1, \ldots, m
\end{align*}
\]

- The minimization is over \(x\) and \(y_j, j = 0, 1, \ldots, m\)
- Convexity preserved
Epigraph Form

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g(x) \leq 0, \ Ax = b
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\text{minimize} & \quad w \\
\text{subject to} & \quad f(x) \leq w \\
& \quad g(x) \leq 0, \ Ax = b
\end{align*}
\]

- The minimization is over \( x \in \mathbb{R}^n \) and \( w \in \mathbb{R} \)
- Is convexity preserved?
Polyhedral Objective Example

\begin{align*}
\text{minimize} & \quad \max\{a_1^T x + b_1, \ldots, a_m^T x + b_m\} \\
\text{subject to} & \quad B^T x \leq d
\end{align*}

- Using the epigraph form of \( f \), the problem is equivalent to an LP

\begin{align*}
\text{minimize} & \quad w \\
\text{subject to} & \quad a_j^T x + b_j \leq w, \quad j = 1, \ldots, m \\
& \quad B^T x \leq d
\end{align*}
Minimizing over some Variables

minimize \( f(x_1, x_2) \)
subject to \( g_j(x_1) \leq 0, \quad j = 1, \ldots, \tilde{m} \)
\( g_j(x_2) \leq 0, \quad j = \tilde{m} + 1, \ldots, m \)

is equivalent to

minimize \( F(x_2) \)
subject to \( g_j(x_2) \leq 0, \quad j = \tilde{m} + 1, \ldots, m \)

where

\[
F(x_2) = \inf_{x_1: g_1(x_1) \leq 0, \ldots, g_{\tilde{m}}(x_1) \leq 0} f(x_1, x_2)
\]