Lecture 4

Closed Functions

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Outline

- Log-Transformation
- Jensen's Inequality
- Level Sets
- Closed Functions
- Convexity and Continuity

Log-Transformation of Variables

Useful for transforming a nonconvex function to a convex one

A *posynomial* is a function of $y_1 > 0, \ldots, y_n > 0$ of the form

$$g(y_1, \ldots, y_n) = b y_1^{a_1} \cdots y_n^{a_n}$$

with scalars $b > 0$ and $a_i > 0$ for all i.

- A posynomial need not be convex
- Log-transformation of variables

$$x_i = \ln y_i \text{ for all } i$$

- We have a convex function

$$f(x) = b e^{a_1 x_1} \cdots e^{a_n x_n} = b e^{a' x}$$

Log-Transformation of Functions

Replacing f with $\ln f$ [when $f(x) > 0$ over $\mathrm{dom} f$]
Useful for:

- Transforming non-separable functions to separable ones

  Example: (Geometric Mean) $f(x) = \left(\Pi_{i=1}^{n} x_i\right)^{1/n}$ for x with $x_i > 0$ for all i is non-separable. Using $F(x) = \ln f(x)$, we obtain a separable F,

$$F(x) = \frac{1}{n} \sum_{i=1}^{n} \ln x_i$$

- Separable structure of objective function is advantageous in distributed optimization

General Convex Inequality

Basic convex inequality: For a convex f, we have for $x, y \in \mathrm{dom} f$ and $\alpha \in (0, 1)$,

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$$

General convex inequality: For a convex f and any convex combination of the points in $\mathrm{dom} f$, we have

$$f\left(\sum_{i=1}^{m} \alpha_i x_i\right) \leq \sum_{i=1}^{m} \alpha_i f(x_i)$$

($x_i \in \mathrm{dom} f$ and $\alpha_i > 0$ for all i, $\sum_{i=1}^{m} \alpha_i = 1$, $m > 0$ integer)

A convex combination $\sum_{i=1}^{m} \alpha_i x_i$ can be viewed as the expectation of a random vector z having outcomes $z = x_i$ with probability α_i

Jensen's Inequality

General convex inequality can be interpreted as:

- For a convex f and a (finite) discrete random variable z with outcomes $z_i \in \mathrm{dom} f$, we have

$$f(\mathrm{E}z) \leq \mathrm{E}\left[f(z)\right]$$

General Jensen's inequality: The above relation holds for a convex f and, a random variable z with outcomes in $\mathrm{dom} f$ and a finite expectation $\mathrm{E}z$

Level Sets

Def. Given a scalar $c \in \mathbb{R}$ and a function $f : \mathbb{R}^n \to \mathbb{R}$, a (lower) *level set* of f associated with c is given by

$$L_c(f) = \{x \in \mathrm{dom} f \mid f(x) \le c\}$$

Examples: $f(x) = \|x\|^2$ for $x \in \mathbb{R}^n$, $\qquad f(x_1, x_2) = e^{x_1}$

- Every level set of a convex function is convex
- Converse is false: Consider $f(x) = -e^x$ for $x \in \mathbb{R}$

Def. A function g is **concave** when $-g$ is convex

- Every (upper) level set of a concave function is convex

Convex Extended-Value Functions

- The definition of convexity that we have used thus far is applicable to functions mapping from a subset of $\mathbb{R}^n$ to $\mathbb{R}^n$. It does not apply to extended-value functions mapping from a subset of $\mathbb{R}^n$ to the extended set $\mathbb{R} \cup \{-\infty, +\infty\}$.
- The general definition of convexity relies on the epigraph of a function
- Let f be a function taking values in $\mathbb{R} \cup \{-\infty, +\infty\}$. The **epigraph** of f is the set given by

$$\mathrm{epi} f = \{(x, w) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq w \text{ for some } x \in \mathbb{R}^n\}.$$

- **Def.** The extended-value function f is convex when its epigraph $\mathrm{epi} f$ is convex set (in $\mathbb{R}^n \times \mathbb{R}$).
- This definition coincides with the one we have used for a function f with values in $\mathbb{R}$.

Closed Functions

Def. A function f is **closed** if its epigraph $\mathrm{epi} f$ is a closed set in $\mathbb{R}^n \times \mathbb{R}$, i.e.,

for every sequence $\{(x_k, w_k)\} \subset \mathrm{epi} f$ converging to some $(\hat{x}, \hat{w})$, we have $(\hat{x}, \hat{w}) \in epif$

Examples
Affine functions are closed $[f(x) = a'x + b]$
Quadratic functions are closed $[f(x) = x'Px + a'x + b]$
Continuous functions are closed

- A class of closed functions is larger than the class of continuous functions
- For example
  $f(x) = 0$ for $x = 0$, $f(x) = 1$ for $x > 0$, and $f(x) = +\infty$ otherwise
  This f is closed but not continuous

Closed Function Properties

Lower-Semicontinuity Def. A function f is *lower-semicontinuous at a given vector* x_0 if for every sequence $\{x_k\}$ converging to x_0, we have

$$f(x_0) \leq \liminf_{k\to 0} f(x_k)$$

We say that f is *lower-semicontinuous over a set* X if f is lower-semicontinuous at every $x \in X$

Th. For a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ the following statements are equivalent

(i) f is closed

(ii) Every level set of f is closed

(iii) f is lower-semicontinuous (l.s.c.) over $\mathbb{R}^n$

Proof of Theorem

(i)⇒(ii) Let c be any scalar and consider $L_c(f)$. If $L_c(f) = \emptyset$, then $L_c(f)$ is closed. Suppose now that $L_c(f) \neq \emptyset$. **Pick** $\{x_k\} \subset L_c(f)$ **such that** $x_k \to \bar{x}$ **for some** $\bar{x} \in \mathbb{R}^n$. We have $f(x_k) \leq c$ for all k, implying that $(x_k, c) \in \text{epi}f$ for all k. Since $(x_k, c) \to (\bar{x}, c)$ and $\text{epi}f$ is closed, it follows that $(\bar{x}, c) \in \text{epi}f$. Consequently $f(\bar{x}) \leq c$, **showing that** $\bar{x} \in L_c(f)$.

(ii)⇒(iii) **Let** $x_0 \in \mathbb{R}^n$ **be arbitrary and let** $\{x_k\}$ **be a sequence such that** $x_k \to x_0$. To arrive at a contradiction, assume that f is not l.s.c. at x_0, i.e.,

$$\liminf_{k\to\infty} f(x_k) < f(x_0)$$

Then, there exist a scalar γ and a subsequence $\{x_k\}_{\mathcal{K}} \subset \{x_k\}$ such that

$$f(x_k) \leq \gamma < f(x_0) \text{ for all } k \in \mathcal{K}$$

yielding that $\{x_k\}_{\mathcal{K}} \subset L_\gamma(f)$. Since $x_k \to x_0$ and the set $L_\gamma(f)$ is closed, it follows that $x_0 \in L_\gamma(f)$. Hence, $f(x_0) \leq \gamma$ - a contradiction. Thus, **we** must **have**

$$f(x_0) \leq \liminf_{k\to\infty} f(x_k)$$

Proof of Theorem - continues

(iii)⇒(i) **To arrive at a contradiction assume that** $\mathrm{epi} f$ **is not closed**. Then, there exists a sequence $\{(x_k, w_k)\} \subset \mathrm{epi} f$ such that

$$(x_k, w_k) \to (\bar{x}, \bar{w}) \quad \text{and} \quad (\bar{x}, \bar{w}) \notin \mathrm{epi} f \qquad .$$

Since $(x_k, w_k) \in \mathrm{epi} f$ for all k, we have

$$f(x_k) \le w_k \qquad \text{for all } k$$

Taking the limit inferior as $k \to \infty$, and using $w_k \to \bar{w}$, we obtain

$$\liminf_{k\to\infty} f(x_k) \le \lim_{k\to\infty} w_k = \bar{w}.$$

Since $(\bar{x}, \bar{w}) \notin \mathrm{epi} f$, we have $f(\bar{x}) > \bar{w}$, implying that

$$\liminf_{k\to\infty} f(x_k) \le \bar{w} < f(\bar{x})$$

On the other hand, because $x_k \to \bar{x}$, and f is l.s.c. at $\bar{x}$, we have

$$f(\bar{x}) \le \liminf_{k\to\infty} f(x_k)$$

- a contradiction. Hence, $\mathrm{epi} f$ must be closed.

Operations Preserving Closedness

- *Positive Scaling*
  For a closed function $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ and $\lambda > 0$, the function
  $$g(x) = \lambda f(x) \text{ is closed}$$

- *Sum*
  For closed functions $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$, $i = 1, \ldots, m$, the sum
  $$g(x) = \sum_{i=1}^m f_i(x) \text{ is closed}$$

- *Composition with Affine Mapping*
  For an $m \times n$ matrix A, a vector $b \in \mathbb{R}^m$, and a closed function $f : \mathbb{R}^m \to \mathbb{R} \cup \{-\infty, +\infty\}$, the function
  $$g(x) = f(Ax + b) \text{ is closed}$$

This generalizes to a Composition with Continuous Mapping

Pointwise Supremum

For a collection of closed functions $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ over an arbitrary index set I, the function

$$g(x) = \sup_{i \in I} f_i(x) \quad \text{is closed}$$

- Example: Piecewise-linear function (polyhedral function)

$$f(x) = \max\{a_1' x + b_1, \ldots, a_m' x + b_m\} \quad \text{is closed}$$

where $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for all i

Convexity and Continuity

- Let $f : \mathbb{R}^n \to \mathbb{R}$ with $\mathrm{dom} f = \mathbb{R}^n$.

If f is convex, then f is continuous over $\mathbb{R}^n$

  - In this case, a convex f is l.s.c

Theorem Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and such that $\mathrm{int}(\mathrm{dom} f) \neq \emptyset$. Then, f is **continuous** over $\mathrm{int}(\mathrm{dom} f)$.

- In this case, a convex f need not be l.s.c over $\mathbb{R}^n$

- Example:

$$f(x) = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{for } x > 0 \\ +\infty & \text{otherwise} \end{cases}$$

Proof of the Theorem

- Using the translation if necessary, we may assume without loss of generality that the origin is in the interior of the domain of f.

- It is sufficient to show that f is continuous at the origin

- By scaling the unit box if necessary, we may assume without loss of generality that the unit box $\{x \in \mathbb{R}^n \mid \|x\|_\infty \le 1\}$ is contained in $\mathrm{dom} f$

- Let v_i, $i \in \mathcal{I} = \{1, \ldots, 2^n\}$ be vertices of the unit box (i.e., each v_i has entries 1 or -1). The unit box can be viewed as a simplex generated by these vertices, i.e.,

  every x with $\|x\|_\infty \le 1$ is a convex combination of vertices $v_i, i \in \mathcal{I}$

or equivalently: every x with $\|x\|_\infty \le 1$ is given by

$$x = \sum_{i \in \mathcal{I}} \alpha_i v_i \qquad \text{with } \alpha_i \ge 0 \text{ and } \sum_{i \in \mathcal{I}} \alpha_i = 1$$

- Note that by convexity of f, we have

$$f(x) \le \max_{i \in \mathcal{I}} f(v_i) = M \tag{1}$$

- Let $x_k \to 0$ and assume that $x_k \ne 0$ for all k
- We introduce $y_k = \frac{x_k}{\|x_k\|_\infty}$ and $z_k = \frac{-x_k}{\|x_k\|_\infty}$
- Note that we can write 0 as a convex combination of y_k and z_k, as follows

$$0 = \frac{1}{\|x_k\|_\infty + 1} x_k + \frac{\|x_k\|_\infty}{\|x_k\|_\infty + 1} z_k \qquad \text{for all } k$$

- By convexity of f it follows that

$$f(0) \leq \frac{1}{\|x_k\|_\infty + 1} f(x_k) + \frac{\|x_k\|_\infty}{\|x_k\|_\infty + 1} f(z_k) \qquad \text{for all } k$$

- By letting $k \to \infty$ and using Eq. (1), we have

$$f(0) \leq \liminf_{k \to 0} f(x_k) \tag{2}$$

- Note that we can write $x_k = (1 - \|x_k\|_\infty)\, 0 + \|x_k\|_\infty\, y_k$ for all k

- By using convexity, we obtain

$$f(x_k) \leq (1 - \|x_k\|_\infty)\, f(0) + \|x_k\|_\infty\, f(y_k)$$

- Taking the limsup as $k \to \infty$ and using Eq. (1), we see that

$$\limsup_{k\to\infty} f(x_k) \leq f(0)$$

- From this relation and Eq. (2), we see that

$$\lim_{k\to\infty} f(x_k) = f(0),$$

showing that f is continuous at 0