Outline

• Log-Transformation
• Jensen’s Inequality
• Level Sets
• Closed Functions
• Convexity and Continuity
Log-Transformation of Variables

Useful for transforming a nonconvex function to a convex one

A posynomial is a function of $y_1 > 0, \ldots, y_n > 0$ of the form

$$g(y_1, \ldots, y_n) = b y_1^{a_1} \cdots y_n^{a_n}$$

with scalars $b > 0$ and $a_i > 0$ for all $i$.

- A posynomial need not be convex
- Log-transformation of variables
  $$x_i = \ln y_i \text{ for all } i$$
- We have a convex function
  $$f(x) = b e^{a_1 x_1} \cdots e^{a_n x_n} = b e^{a' x}$$
Log-Transformation of Functions

Replacing $f$ with $\ln f$ [when $f(x) > 0$ over $\text{dom} f$]

Useful for:

- Transforming non-separable functions to separable ones

*Example*: (Geometric Mean) $f(x) = (\prod_{i=1}^{n} x_i)^{1/n}$ for $x$ with $x_i > 0$ for all $i$ is non-separable. Using $F(x) = \ln f(x)$, we obtain a separable $F$,

$$F(x) = \frac{1}{n} \sum_{i=1}^{n} \ln x_i$$

- Separable structure of objective function is advantageous in distributed optimization
**General Convex Inequality**

*Basic convex inequality:* For a convex $f$, we have for $x, y \in \text{dom} f$ and $\alpha \in (0, 1)$,

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$$

*General convex inequality:* For a convex $f$ and any convex combination of the points in $\text{dom} f$, we have

$$f \left( \sum_{i=1}^{m} \alpha_i x_i \right) \leq \sum_{i=1}^{m} \alpha_i f(x_i)$$

($x_i \in \text{dom} f$ and $\alpha_i > 0$ for all $i$, $\sum_{i=1}^{m} \alpha_i = 1$, $m > 0$ integer)

A convex combination $\sum_{i=1}^{m} \alpha_i x_i$ can be viewed as the expectation of a random vector $z$ having outcomes $z = x_i$ with probability $\alpha_i$
Jensen’s Inequality

General convex inequality can be interpreted as:

- For a convex $f$ and a (finite) discrete random variable $z$ with outcomes $z_i \in \text{dom } f$, we have

$$f(\mathbb{E}z) \leq \mathbb{E}[f(z)]$$

*General Jensen’s inequality:* The above relation holds for a convex $f$ and, a random variable $z$ with outcomes in $\text{dom } f$ and a finite expectation $\mathbb{E}z$
Level Sets

**Def.** Given a scalar $c \in \mathbb{R}$ and a function $f : \mathbb{R}^n \to \mathbb{R}$, a (lower) level set of $f$ associated with $c$ is given by

$$L_c(f) = \{ x \in \text{dom} f \mid f(x) \leq c \}$$

Examples: $f(x) = \|x\|^2$ for $x \in \mathbb{R}^n$, $f(x_1, x_2) = e^{x_1}$

- Every level set of a convex function is convex

- Converse is false: Consider $f(x) = -e^x$ for $x \in \mathbb{R}$

**Def.** A function $g$ is **concave** when $-g$ is convex

- Every (upper) level set of a concave function is convex
Convex Extended-Value Functions

- The definition of convexity that we have used thus far is applicable to functions mapping from a subset of $\mathbb{R}^n$ to $\mathbb{R}^n$. It does not apply to extended-value functions mapping from a subset of $\mathbb{R}^n$ to the extended set $\mathbb{R} \cup \{-\infty, +\infty\}$.
- The general definition of convexity relies on the epigraph of a function.
- Let $f$ be a function taking values in $\mathbb{R} \cup \{-\infty, +\infty\}$. The epigraph of $f$ is the set given by

$$\text{epi} f = \{(x, w) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq w \text{ for some } x \in \mathbb{R}^n\}.$$ 

- **Def.** The extended-value function $f$ is convex when its epigraph $\text{epi} f$ is convex set (in $\mathbb{R}^n \times \mathbb{R}$).
- This definition coincides with the one we have used for a function $f$ with values in $\mathbb{R}$.
Closed Functions

**Def.** A function $f$ is closed if its epigraph $\text{epi}\ f$ is a closed set in $\mathbb{R}^n \times \mathbb{R}$, i.e.,

for every sequence $\{(x_k, w_k)\} \subset \text{epi}\ f$ converging to some $(\hat{x}, \hat{w})$, we have $(\hat{x}, \hat{w}) \in \text{epi}\ f$

**Examples**

Affine functions are closed $[f(x) = a'x + b]$

Quadratic functions are closed $[f(x) = x'Px + a'x + b]$

Continuous functions are closed

- A class of closed functions is larger than the class of continuous functions
- For example

  $f(x) = 0$ for $x = 0$, $f(x) = 1$ for $x > 0$, and $f(x) = +\infty$ otherwise

  This $f$ is closed but not continuous
Closed Function Properties

**Lower-Semicontinuity Def.** A function \( f \) is *lower-semicontinuous at a given vector* \( x_0 \) if for every sequence \( \{x_k\} \) converging to \( x_0 \), we have
\[
f(x_0) \leq \lim \inf_{k \to 0} f(x_k)
\]

We say that \( f \) is *lower-semicontinuous over a set* \( X \) if \( f \) is lower-semicontinuous at every \( x \in X \)

**Th.** For a function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\} \) the following statements are equivalent

(i) \( f \) is closed

(ii) Every level set of \( f \) is closed

(iii) \( f \) is lower-semicontinuous (l.s.c.) over \( \mathbb{R}^n \)
Proof of Theorem

(i)⇒(ii) Let $c$ be any scalar and consider $L_c(f)$. If $L_c(f) = \emptyset$, then $L_c(f)$ is closed. Suppose now that $L_c(f) \neq \emptyset$. Pick $\{x_k\} \subset L_c(f)$ such that $x_k \to \bar{x}$ for some $\bar{x} \in \mathbb{R}^n$. We have $f(x_k) \leq c$ for all $k$, implying that $(x_k, c) \in \text{epi} f$ for all $k$. Since $(x_k, c) \to (\bar{x}, c)$ and $\text{epi} f$ is closed, it follows that $(\bar{x}, c) \in \text{epi} f$. Consequently $f(\bar{x}) \leq c$, showing that $\bar{x} \in L_c(f)$.

(ii)⇒(iii) Let $x_0 \in \mathbb{R}^n$ be arbitrary and let $\{x_k\}$ be a sequence such that $x_k \to x_0$. To arrive at a contradiction, assume that $f$ is not l.s.c. at $x_0$, i.e.,

$$\lim \inf_{k \to \infty} f(x_k) < f(x_0)$$

Then, there exist a scalar $\gamma$ and a subsequence $\{x_k\}_K \subset \{x_k\}$ such that $f(x_k) \leq \gamma < f(x_0)$ for all $k \in K$ yielding that $\{x_k\}_K \subset L_{\gamma}(f)$. Since $x_k \to x_0$ and the set $L_{\gamma}(f)$ is closed, it follows that $x_0 \in L_{\gamma}(f)$. Hence, $f(x_0) \leq \gamma$ - a contradiction. Thus, we must have

$$f(x_0) \leq \lim \inf_{k \to \infty} f(x_k)$$
Proof of Theorem - continues

(iii) $\Rightarrow$ (i) To arrive at a contradiction assume that $\text{epi } f$ is not closed. Then, there exists a sequence $\{(x_k, w_k)\} \subset \text{epi } f$ such that $(x_k, w_k) \to (\bar{x}, \bar{w})$ and $(\bar{x}, \bar{w}) \notin \text{epi } f$. Since $(x_k, w_k) \in \text{epi } f$ for all $k$, we have

$$f(x_k) \leq w_k \quad \text{for all } k$$

Taking the limit inferior as $k \to \infty$, and using $w_k \to \bar{w}$, we obtain

$$\liminf_{k \to \infty} f(x_k) \leq \lim_{k \to \infty} w_k = \bar{w}.$$ 

Since $(\bar{x}, \bar{w}) \notin \text{epi } f$, we have $f(\bar{x}) > \bar{w}$, implying that

$$\liminf_{k \to \infty} f(x_k) \leq \bar{w} < f(\bar{x})$$

On the other hand, because $x_k \to \bar{x}$, and $f$ is l.s.c. at $\bar{x}$, we have

$$f(\bar{x}) \leq \liminf_{k \to \infty} f(x_k)$$

- a contradiction. Hence, $\text{epi } f$ must be closed.
Operations Preserving Closedness

• **Positive Scaling**
  For a closed function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\} \) and \( \lambda > 0 \), the function \( g(x) = \lambda f(x) \) is closed

• **Sum**
  For closed functions \( f_i : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}, i = 1, \ldots, m \), the sum \( g(x) = \sum_{i=1}^{m} f_i(x) \) is closed

• **Composition with Affine Mapping**
  For an \( m \times n \) matrix \( A \), a vector \( b \in \mathbb{R}^m \), and a closed function \( f : \mathbb{R}^m \to \mathbb{R} \cup \{-\infty, +\infty\} \), the function \( g(x) = f(Ax + b) \) is closed

  This generalizes to a Composition with Continuous Mapping
Pointwise Supremum

For a collection of closed functions $f_i : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ over an arbitrary index set $I$, the function

$$g(x) = \sup_{i \in I} f_i(x)$$

is closed.

• Example: Piecewise-linear function (polyhedral function)

$$f(x) = \max\{a'_1 x + b_1, \ldots, a'_m x + b_m\}$$

is closed

where $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for all $i$
Convexity and Continuity

- Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ with $\text{dom} f = \mathbb{R}^n$.

  **If $f$ is convex, then $f$ is continuous over $\mathbb{R}^n$**

- In this case, a convex $f$ is l.s.c.

**Theorem**  Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex and such that $\text{int}(\text{dom} f) \neq \emptyset$. Then, $f$ is **continuous** over $\text{int}(\text{dom} f)$.

- In this case, a convex $f$ need not be l.s.c over $\mathbb{R}^n$

- Example:

  $$f(x) = \begin{cases} 
  1 & \text{for } x = 0 \\
  0 & \text{for } x > 0 \\
  +\infty & \text{otherwise}
  \end{cases}$$
Proof of the Theorem

• Using the translation if necessary, we may assume without loss of generality that the origin is in the interior of the domain of $f$.

• It is sufficient to show that $f$ is continuous at the origin.

• By scaling the unit box if necessary, we may assume without loss of generality that the unit box \( \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq 1\} \) is contained in $\text{dom} f$.

• Let $v_i, i \in \mathcal{I} = \{1, \ldots, 2^n\}$ be vertices of the unit box (i.e., each $v_i$ has entries 1 or $-1$). The unit box can be viewed as a simplex generated by these vertices, i.e.,

$$
\text{every } x \text{ with } \|x\|_\infty \leq 1 \text{ is a convex combination of vertices } v_i, i \in \mathcal{I}.
$$
or equivalently: every $x$ with $\|x\|_\infty \leq 1$ is given by

$$x = \sum_{i \in I} \alpha_i v_i \quad \text{with } \alpha_i \geq 0 \text{ and } \sum_{i \in I} \alpha_i = 1$$

- Note that by convexity of $f$, we have

$$f(x) \leq \max_{i \in I} f(v_i) = M \quad (1)$$

- Let $x_k \to 0$ and assume that $x_k \neq 0$ for all $k$

- We introduce $y_k = \frac{x_k}{\|x_k\|_\infty}$ and $z_k = \frac{-x_k}{\|x_k\|_\infty}$

- Note that we can write 0 as a convex combination of $y_k$ and $z_k$, as follows

$$0 = \frac{1}{\|x_k\|_\infty + 1} x_k + \frac{\|x_k\|_\infty}{\|x_k\|_\infty + 1} z_k \quad \text{for all } k$$
• By convexity of $f$ it follows that

$$f(0) \leq \frac{1}{\|x_k\|_\infty + 1} f(x_k) + \frac{\|x_k\|_\infty}{\|x_k\|_\infty + 1} f(z_k) \quad \text{for all } k$$

• By letting $k \to \infty$ and using Eq. (1), we have

$$f(0) \leq \liminf_{k \to 0} f(x_k) \quad (2)$$

• Note that we can write $x_k = (1 - \|x_k\|_\infty) 0 + \|x_k\|_\infty y_k$ for all $k$

• By using convexity, we obtain

$$f(x_k) \leq (1 - \|x_k\|_\infty) f(0) + \|x_k\|_\infty f(y_k)$$
• Taking the limsup as \( k \to \infty \) and using Eq. (1), we see that
\[
\limsup_{k \to \infty} f(x_k) \leq f(0)
\]

• From this relation and Eq. (2), we see that
\[
\lim_{k \to \infty} f(x_k) = f(0),
\]

showing that \( f \) is continuous at 0