Lecture 20

Methods for Dual Problems

November 11, 2008

Outline

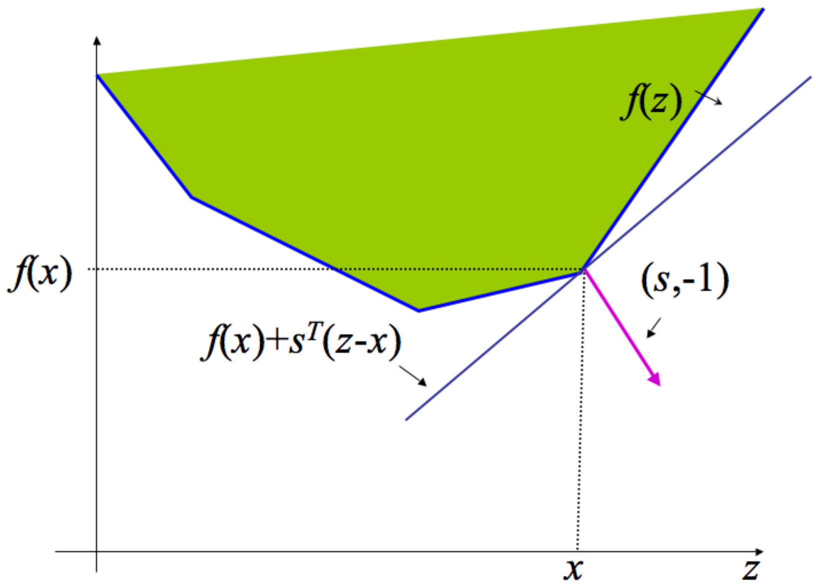
- Interpretation of Polyak's stepsize
- Convergence and Convergence Rate
- Subgradient Methods for Dual Problems

Interpretation of Polyak's stepsize

A vector s is a subgradient of a convex function $f : \mathbb{R}^n \mapsto \mathbb{R}$ at $\hat{x} \in \operatorname{dom} f$ when **subgradient inequality holds**

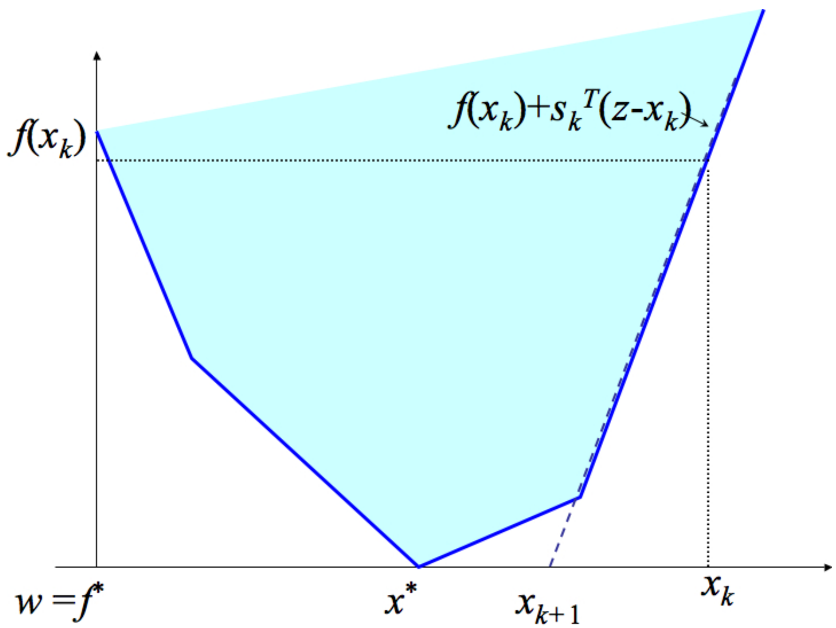
$$f(z) \geq f(\hat{x}) + s^T(z - \hat{x}) \qquad \text{for all } z \in \operatorname{dom} f$$

- We have interpreted the subgradient inequality in terms of **a hyperplane in $\mathbb{R}^{n+1}$ supporting the epigraph $\operatorname{epi} f$ at $(\hat{x}, f(\hat{x}))$**



- Polyak stepsize can be interpreted by looking at the projection of the epigraph and the hyperplane on the set

$$\{(x, w) \mid x \in \mathbb{R}^n,\ w = f^*\}$$



- The projection of the hyperplane is given by

$$\tilde{H} = \{(z, w) \in \mathbb{R}^{n+1} \mid f(\hat{x}) + s^T(z - \hat{x}) = f^*,\ w = f^*\}$$

- By looking only at x-variables (since $w = f^*$), at $\widehat{x} = x_k$, the resulting hyperplane in the reduced space becomes

$$H = \{z \in \mathbb{R}^n \mid f(x_k) + s_k^T(z - x_k) = f^*\}$$

- With Polyak's stepsize, the iterate x_{k+1} is the projection of x_k on H
- To see that note that the projection of x_k on H can be determined by looking at the intersection of the ray $\{z \mid z = x_k + ts_k,\ t \geq 0\}$ with H, which gives

$$f(x_k) + t^*\|s_k\|^2 = f^*.$$

  Solving for t^* yields $t^* = \frac{f^* - f(x_k)}{\|s_k\|^2}$.
- The next iterate is

$$x_{k+1} = x_k + t^* s_k = x_k - \frac{f(x_k) - f^*}{\|s_k\|^2}\, s_k$$

  The Polyak stepsize $\alpha_k = \frac{f(x_k) - f^*}{\|s_k\|^2}$ is equal to the distance from x_k to the hyperplane H, i.e., $\alpha_k = |t^*|$.

Convergence Rate

- The convergence rate of the subgradient method with Polyak's stepsize is linear (at best)
  - For a function f with **sharp minima**, i.e., such that for some $\eta > 0$

$$f(x) - f^* \geq \eta\, dist(x, X^*) \qquad \text{for all } x$$

  - **The rate is linear** (HW8)

$$\|x_k - \tilde{x}^*\| \leq c^k\, \|x_0 - \tilde{x}^*\| \qquad \text{for all } k \geq 0$$

  where $\tilde{x}^* \in X^*$ is the limit point of $\{x_k\}$ and

$$c = \sqrt{1 - \frac{\eta^2}{L^2}}$$

  and L is an upper bound on the subgradient norms $\|s_k\|$

- The rate is important for general understanding of the method
- It is rare that we can take advantage of this result in practice

Comments

- Subgradient methods considered so far:
    - Use any subgradient that is available at a given iterate
    - Simple for implementation
    - Convergence rate is at best linear
    - Useful in large-scale and decentralized computations
    - **Main criticism**: There are **no general stopping rules**
        - A specific criteria have been designed within particular applications
- Alternative methods exist: **Bundle methods**
    - Use a carefully selected subgradient at a given iterate
    - More sophisticated for implementation
    - There is a general stopping criteria