Revenue Optimal Auction for Single-Minded Buyers

Vineet Abhishek and Bruce Hajek

Abstract—We study the problem of characterizing revenue optimal auctions for single-minded buyers. We identify revenue optimal auctions with a simple structure, if the conditional distribution of any buyer’s valuation is nondecreasing, in the hazard rates ordering of probability distributions, as a function of the bundle the buyer is interested in. The revenue optimal auction is given by the solution of a maximum weight independent set problem. We provide a novel graphical construction of the virtual valuations.

I. INTRODUCTION

Consider $N$ buyers competing for a certain set of items offered by a seller. A buyer has a value for each combination of the items (henceforth bundle) that he is interested in. This is the maximum price that he is willing to pay for the bundle and is known accurately only to him. The seller’s objective is to maximize his revenue from the sale. The seller’s task is complicated because a buyer can misreport the values of the bundles he is interested in. Combinatorial auctions (henceforth CAs) offer a solution. CAs allow buyers to compete for any bundle of items. The allocation and the payments are determined by the competition among the buyers. However, the inherent problems of CAs limit their appeal.

Often there are complementarities among the items - a buyer can have a higher value for a bundle as a whole than the sum of values of the parts of the bundle. Different buyers may have different forms of complementarity. Because of complementarity, allocation of items in a CA requires solving a hard combinatorial optimization problem. Moreover, theoretical results on revenue maximization (also referred to as optimality) are known only under simple settings.

This paper aims to address the issues of (1) dealing with complementarity, and (2) maximizing revenue, for CAs. An extreme case of complementarity is when buyers are single minded [1]. Here, each buyer is interested only in a specific bundle and has a value for the same. Any allocation of items to a buyer that does not contain his desired bundle has zero value for him. Both the bundle that a buyer is interested in and its value are his private information. While the single-minded buyers model is a simplifying assumption, no general result on revenue maximization is known even for this extreme case. Here a buyer has two dimensions for misreporting his preference - the bundle he is interested in, and the value of the bundle. Most of the existing literature on revenue optimal auctions studies problems which are one dimensional - a buyer has only one real number for misreporting his preference, e.g., Myerson’s single item auction [2], the single-parameter buyers model described in [3], and single-minded buyers with known bundles [4]. Thus, the single-minded buyer model can be thought of as an initial step towards solving the general CA problems. Also, see [1] for some real examples where buyers are single minded. Hence, we focus on the CAs with single-minded buyers.

In addition, we take a departure from the continuous variable models of economics and assume that the set of possible values that a buyer can have for his bundle is finite. This is clearly relevant from the implementation point of view. A related work is [5] where a Bayesian optimal auction, when buyers’ valuation sets are finite, is characterized. However, [5] deals only with single item auctions.

We make the following contributions in this paper. We modify and extend the framework of [5] for multiple item auctions with single-minded buyers. We then find a sufficient condition under which a revenue optimal auction can be characterized for single-minded buyers. This sufficient condition is the monotonicity of the conditional distribution of any buyer’s valuation, in the hazard rate ordering, as a function of the bundle the buyer is interested in. An interpretation of this condition is as follows: if there are two bundles where one contains the other, then the larger bundle is likely to have a higher value. Such monotonicity property is intuitive for single-minded buyers. A similar hazard rate ordering condition appears in [6] in the context of dynamic knapsack problem. However, [6] (as well as [4]) assumes that the distributions also satisfy Myerson’s nonintuitive regularity assumption [2]. An important contribution of our paper is to show that such assumption is unnecessary. We present an algorithm for optimal auction as a solution of a maximum weight independent set problem, where weights are an appropriate mapping of buyers’ valuations, called virtual valuations. We provide a novel graphical construction of the virtual valuations.

The rest of the paper is organized as follows. Section II outlines our model, definitions, and notation. Section III characterizes an optimal auction for single-minded buyers, while Section IV describes some of its important properties. We conclude in Section V.

II. MODEL AND NOTATION

Consider $N$ buyers competing for $S$ items that a seller wants to sell. The set of buyers is denoted by $N \triangleq \{1, 2, \ldots, N\}$, and the set of items for sale is denoted by $S$. Buyers are single minded - each buyer $n$ is interested only in a specific
bundle $b_n^* \in 2^S$ and has a value $v_n^*$ for any bundle $b_n$ such that $b_n \supseteq b_n^*$, while he has zero value for any other bundle. Here $2^S$ denotes the power set of $S$. Notice that single-minded buyers enjoy free disposal of the items. We refer to the tuple $(b_n, v_n)$ as the type of buyer $n$. The type of a buyer is known only to him and constitutes his private information.

For each buyer $n$, the seller and the other buyers have imperfect information about his type; they describe the bundle that buyer $n$ is interested in by a random set $B_n$, and its value by a discrete random variable $X_n$. The random set $B_n$ takes values from the collection $B_n \subseteq 2^S$, where $B_n$ is the collection of all bundles that buyer $n$ can possibly be interested in. The random variable $X_n$ is assumed to take values from the set $X_n \triangleq \{x_1^n, x_2^n, \ldots, x_{K_n}^n\}$ of cardinality $K_n$, where $0 \leq x_1^n < x_2^n < \ldots < x_{K_n}^n$. The joint probability distribution of $B_n$ and $X_n$ is common knowledge. Let $p(b_n) \triangleq \mathbb{P}[B_n = b_n]$, and $p(x_n^n | b_n) \triangleq \mathbb{P}[X_n = x_n^n | B_n = b_n]$. Assume that $p(b_n) > 0$ and $p(x_n^n | b_n) > 0$ for all $n \in \mathcal{N}$, $b_n \in B_n$, and $1 \leq i \leq K_n$. Note that $(b_n, v_n)$ can be interpreted as a specific realization of the random variables $(B_n, X_n)$. Let $Y_n \triangleq (B_n, X_n)$ be the random vector describing the type of buyer $n$. Random vectors $[Y_n]_{n \in \mathcal{N}}$ are assumed to be independent.

Denote a typical reported type (henceforth bid) of a buyer $n$ by $(b_n, v_n)$, where $b_n \in B_n$, and $v_n \in X_n$. Define the vector of bids as $(b, v)$, where $b \triangleq (b_1, b_2, \ldots, b_N)$ is the vector of reported bundles, and $v \triangleq (v_1, v_2, \ldots, v_N)$ is the vector of reported values. The seller can only allocate the items to a set of buyers whose reported bundles are disjoint. Given $b$, define $A(b)$ as follows:

$$A(b) \triangleq \{ A \subseteq \mathcal{N} : \forall m, n \in A, n \neq m, b_n \cap b_m = \emptyset \}. \quad (1)$$

This is the collection of all subsets of buyers who can be allocated their respective bundles simultaneously. Trivially, $\emptyset \in A(b)$ and $A(b)$ is downward closed; i.e., if $A \in A(b)$ and $B \subseteq A$, then $B \in A(b)$.

Define $B \triangleq (B_1, B_2, \ldots, B_N)$, $X \triangleq (X_1, X_2, \ldots, X_N)$, and $Y \triangleq (Y_1, Y_2, \ldots, Y_N)$. The vector of bids is $b \triangleq (b_1, b_2, \ldots, b_N)$, and the vector of types is $Y \triangleq (Y_1, Y_2, \ldots, Y_N)$. We use $Y$ and $(B, X)$ interchangeably. Let $B \triangleq B_1 \times B_2 \times \ldots \times B_N$ and $X \triangleq X_1 \times X_2 \times \ldots \times X_N$. We use the standard game theoretic notation of $v_{-n} \triangleq (v_1, v_2, v_{n+1}, \ldots, v_N)$ and $v \triangleq (v_n, v_{-n})$. Similar interpretations are used for $b_{-n}, B_{-n}, X_{-n}, Y_{-n}, B_{-n}$, and $X_{-n}$. Henceforth, in any further usage, $b_n, b_{-n}$, and $b$ are always in the sets $B_n, B_{-n}, \text{ and } B$ respectively; and $v_n, v_{-n}$, and $v$ are always in the sets $X_n, X_{-n}, \text{ and } X$ respectively.

### III. Revenue Optimal Auction

In this section, we formally describe the optimal auction problem, formulate the objective and the constraints explicitly, and provide an optimal algorithm for solving the problem. We will be focusing only on the auction mechanisms where buyers are asked to report their types directly (referred to as direct mechanism). By the revelation principle [2], the restriction to direct mechanisms is without any loss of optimality.

#### A. Characterization

An auction mechanism is specified by an allocation rule $\pi: \mathcal{B} \times \mathcal{X} \rightarrow [0, 1]^{2^N}$, and a payment rule $M: \mathcal{B} \times \mathcal{X} \rightarrow \mathbb{R}^N$. Given a bid vector $(b, v)$, the allocation rule $\pi(b, v)$ is a probability distribution over the power set $2^\mathcal{N}$ of $\mathcal{N}$. For each $A \subseteq 2^\mathcal{N}$, $\pi_A(b, v)$ is the probability that the set of buyers $A$ get their reported bundles simultaneously. The allocation must be feasible. Hence, for any $b, v$, and $A \notin A(b)$, we trivially set $\pi_A(b, v) = 0$. Or equivalently, for a given $b$, $\pi$ is a probability distribution on $A(b)$. The payment rule is defined as $M \triangleq (M_1, M_2, \ldots, M_N)$, where $M_n(b, v)$ is the payment (expected payment in case of random allocation) that buyer $n$ makes to the seller when the bid vector is $(b, v)$. Let $Q_n(b, v)$ be the probability that buyer $n$ gets his reported bundle $b_n$ when the bid vector is $(b, v)$; i.e.,

$$Q_n(b, v) \triangleq \sum_{A \subseteq A(b), n \in A} \pi_A(b, v). \quad (2)$$

Buyers are assumed to be risk neutral and have quasilinear payoffs. Given that the type of a buyer $n$ is $(b_n, v_n)$, and the bid vector is $(b, v)$, the payoff (expected payoff in case of random allocation) of the buyer $n$ is:

$$\sigma_n(b, v; b_n, v_n^*) \triangleq Q_n(b, v)1_{(b_n \subseteq b_n^{*})} v_n^{*} - M_n(b, v). \quad (3)$$

The mechanism $(\pi, M)$ and the payoff functions $[\sigma_n]_{n \in \mathcal{N}}$ induce a game of incomplete information among the buyers. We use Bayes-Nash equilibrium (BNE) as the solution concept. The seller’s goal is to design an auction mechanism $(\pi, M)$ to maximize his expected revenue at a BNE of the induced game. Again, using the revelation principle, seller can restrict only to the auctions where truth-telling is a BNE (referred to as incentive compatibility) without any loss of optimality. Assume that the seller cannot force the buyers to participate in an auction and a buyer will voluntarily participate in an auction only if his payoff from participation is nonnegative (referred to as individual rationality). The seller too is assumed to have free disposal of the items and may decide not to sell some or all items for certain bid vectors.

The idea now, as in [2], is to express incentive compatibility and individual rationality as mathematical constraints, and formulate the revenue maximization objective as an optimization problem under these constraints.

For each $n \in \mathcal{N}$, $b_n$, and $v_n$, define the following functions:

$$q_n(b_n, v_n) \triangleq \mathbb{E}[Q_n(b_n, v_n, X_{-n})], \quad (4)$$

$$m_n(b_n, v_n) \triangleq \mathbb{E}[M_n(b_n, v_n, Y_{-n})], \quad (5)$$

$$u_n(b_n, v_n) \triangleq q_n(b_n, v_n) v_n - m_n(b_n, v_n). \quad (6)$$

Here, $q_n(b_n, v_n)$ is the expected probability that buyer $n$ gets his bundle given that he reports his type as $(b_n, v_n)$ while everyone else is truthful. The expectation here is over the type of everyone else; i.e., over $Y_{-n}$. Similarly, $m_n(b_n, v_n)$ is the expected payment that buyer $n$ makes to the seller. $u_n(b_n, v_n)$ is the expected payoff of buyer $n$ given that his true type is
Modified optimal auction problem (MOAP)

\[
\begin{align*}
\text{maximize} & \quad \mathbb{E} \left[ \sum_{n=1}^{N} M_n(Y) \right], \\
\text{subject to} & \quad \text{IC and IR constraints}.
\end{align*}
\]

(9)

Optimal auction problem (OAP)

\[
\begin{align*}
\text{maximize} & \quad \mathbb{E} \left[ \sum_{n=1}^{N} M_n(Y) \right], \\
\text{subject to} & \quad \text{IC and IR constraints}.
\end{align*}
\]

(11)

B. Solution of the MOAP

We now describe an algorithm for solving the MOAP. As mentioned in Section I, this is related to [3], [4], and [5]. Proposition 1 and (14) suggest that a solution of the MOAP can be found by selecting the allocation rule \( \pi \) that assigns nonzero probabilities to the set of buyers in \( \mathcal{A}(b) \) with the maximum total virtual valuations for each bid vector \((b, v)\). If all \( w_n \)'s are regular, then it can be verified that such an allocation rule satisfies the monotonicity condition on the \( q_n \)'s needed by Proposition 1. However, if \( w_n \)'s are not regular, the resulting allocation rule would not necessarily satisfy the required monotonicity condition on the \( q_n \)'s. This problem can be remedied by using another function, \( \pi_n \), called the monotone virtual valuation (henceforth MVV), constructed graphically as follows.

For all \( n \in \mathcal{N}, b_n \), and \( 0 \leq i \leq K_n \), define:

\[
(g_{n,b_n}^{b_n,i}, h_{n,b_n}^{b_n,i}) \equiv \left( \frac{\sum_{j=1}^{K_n} p(x_n^j | b_n)}{p(x_n^i | b_n)}, -x_n^{i+1} + \left( \frac{\sum_{j=1}^{K_n} p(x_n^j | b_n)}{p(x_n^i | b_n)} \right) \right),
\]

(15)

where we use the notational convention of \( \sum_{j=1}^{0} (.) \equiv 0 \), \( x_n^{K_n+1} \equiv 0 \), and \( \sum_{j=K_n+1}^{K_n} (.) \equiv 0 \). Then, \( w_n(b_n, x_n^i) \) is given by the slope of the line joining the point \((g_{n,b_n}^{b_n,i-1}, h_{n,b_n}^{b_n,i-1})\) to the point \((g_{n,b_n}^{b_n,i}, h_{n,b_n}^{b_n,i})\); i.e.,

\[
w_n(b_n, x_n^i) = \frac{h_{n,b_n}^{b_n,i} - h_{n,b_n}^{b_n,i-1}}{g_{n,b_n}^{b_n,i} - g_{n,b_n}^{b_n,i-1}}.
\]

(16)

Find the lower convex hull of the points \([g_{n,b_n}^{b_n,i}, h_{n,b_n}^{b_n,i}]_{0 \leq i \leq K_n}\). Let \( \overline{g}_{n,b_n}^{b_n,i} \) be the point on this convex hull corresponding to \( g_{n,b_n}^{b_n,i} \). Then, \( \overline{w}_n(b_n, x_n^i) \) is defined as the slope of the line joining the point \((g_{n,b_n}^{b_n,i-1}, h_{n,b_n}^{b_n,i-1})\) to the point \((g_{n,b_n}^{b_n,i}, \overline{h}_{n,b_n}^{b_n,i})\); i.e.,

\[
\overline{w}_n(b_n, x_n^i) = \frac{\overline{h}_{n,b_n}^{b_n,i} - h_{n,b_n}^{b_n,i-1}}{g_{n,b_n}^{b_n,i} - g_{n,b_n}^{b_n,i-1}}.
\]

(17)
To find the points \( \{(p_{bi,v}^{b_n,i}, h_{bi,v}^{b_n,i})\} \) graphically, draw vertical lines separated from each other by distances \( p(x_i^1(b_n)), p(x_i^2(b_n)), \ldots, p(x_i^{K_n}(b_n)) \). For each \( 1 \leq i \leq K_n \), join the point \(-x^i\) on the y-axis to the x-axis at \( 1/(\text{sum of probabilities}) \) and call such line as line \( i \). The intersection of line \( i \) with y-axis is the point \((g_{bi,v}^{b_n,0}, h_{bi,v}^{b_n,0})\). The intersection of line \( 2 \) with the first vertical line is the point \((g_{bi,v}^{b_n,1}, h_{bi,v}^{b_n,1})\). Similarly, the intersection of the line \( 3 \) with the second vertical line is the point \((g_{bi,v}^{b_n,2}, h_{bi,v}^{b_n,2})\), and so on. Notice that if \( w_n \) is regular, \( \overline{w}_n \) is equal to \( w_n \).

Figure 1 shows this construction for a typical random variable \( X \) taking four different values \( \{x_1, x_2, x_3, x_4\} \) with corresponding probabilities \( \{p^1, p^2, p^3, p^4\} \). We have dropped the subscripts corresponding to the buyers and the bundle information for the ease of notation. Since \( x_1 < x_2 \), the virtual-valuation function is not regular. Here, the the lower convex hull of the points \((g^i, h^i)\)'s is taken. The slopes of individual segments of this convex hull give the MVV function \( \overline{w} \). This is equivalent to replacing \( w(x^i) \) and \( w(x^2) \) by their weighted mean; i.e., \( \overline{w}(x^i) = \overline{w}(x^2) = (p^1w(x^i) + p^2w(x^2))/(p^1 + p^2) \).

The following lemma is a straightforward consequence of the construction of \( \overline{w}_n \) as the slopes of a convex function:

**Lemma 1:** \( \overline{w}_n(b_n, x_i^1) \leq \overline{w}_n(b_n, x_i^{i+1}) \) for all \( n \in N, b_n, \) and \( 1 \leq i \leq K_n - 1 \).

The next proposition establishes the significance of the MVVs. The proof is a straightforward extension of [5] where a similar result is obtained for single item auctions. Details are omitted because of space constraints.

**Proposition 2:** Given any allocation rule \( \pi \) such that \( Q_n \in N \) and \( [q_n] \in N \), obtained from \( \pi \) by (2) and (4), satisfy \( q_n(b_n, x_i^1) \leq q_n(b_n, x_i^{i+1}) \), for all \( n \in N, b_n, \) and \( 1 \leq i \leq K_n - 1 \). Then,

\[
\mathbb{E} \left[ \sum_{n=1}^{N} Q_n(Y) w_n(Y_n) \right] \leq \mathbb{E} \left[ \sum_{n=1}^{N} Q_n(Y) \overline{w}_n(Y_n) \right] \quad \text{(18)}
\]

Moreover, (18) holds with equality for any \( \pi \) that maximizes \( \sum_{n=1}^{N} Q_n(b, v) \overline{w}_n(b, v_n) \) for each bid vector \( (b, v) \).

The maximum weight algorithm (henceforth MWA) for the MOAP is described in Algorithm 1. The set \( W(b, v) \) is the collection of all feasible subsets of buyers with maximum total MVVs for the given bid vector \( (b, v) \). Since \( A(b) \) is downward closed and \( \emptyset \in A(b) \), no buyer \( n \) with \( \overline{w}_n(b_n, v_n) < 0 \) is included in the set of winners \( W(b, v) \). In step 3 of the MWA, for each \( x_i^1 \leq v_n, \) \( Q_n(b, x_i^1, v_n) \) is computed recursively by treating \( (b, x_i^1, v_n) \) as the input bid vector and repeating steps 1–2.

**Algorithm 1 Maximum weight algorithm (MWA)**

Given a bid vector \( (b, v) \):

1) Compute \( \overline{w}_n(b_n, v_n) \) for each \( n \in N \).

2) Take \( \pi(b, v) \) to be any probability distribution on the collection \( W(b, v) \) defined as:

\[
W(b, v) \triangleq \operatorname{argmax}_{A \in A(b)} \sum_{n \in A} \overline{w}_n(b_n, v_n).
\]

Obtain the set of winners \( W(b, v) \) by sampling from \( W(b, v) \) according to \( \pi(b, v) \).

3) Collect payments given by:

\[
M_n(b, v) = \sum_{i: x_i^1 \leq v_n} (Q_n(b, x_i^1, v_n) - Q_n(b, x_i^{i-1}, v_n) x_i^1),
\]

where \( Q_n \) is given by (2), and \( Q_n(b, x_i^1, v_n) \) is defined as:

\[
Q_n(b, x_i^1, v_n) = \sum_{e \in E} \sum_{w \in W} \pi_e(w) f(b, w, v) - f(b, w, v_n).
\]

**Proposition 3:** The MWA gives a solution of the MOAP.  
**Proof:** This follows from Propositions 1-2, and Lemma 1. Details are omitted because of space constraints.

The MWA can be interpreted as follows. Given a bid vector \( (b, v) \), construct a graph \( G_b(N, E) \) with a node \( n \) for each buyer \( b_n \), and an edge \( e_{v_n,n} \in E \) if \( b_n \cap b_m \neq \emptyset \). Thus, \( G_b \) is the conflict graph of the buyers, where an edge denotes that buyers corresponding to its endpoints cannot be allocated their bundles simultaneously. The collection of all independent sets of this graph is precisely \( A(b) \). Let \( \overline{w}_n(b_n, v_n) \) be the weight of node \( n \). Then the set of winners \( W(b, v) \) is a maximum weight independent set of this graph.

In the subsequent discussion, we will be using the MWA with a deterministic tie-breaking rule (henceforth deterministic MWA). Here, in step 2 of the MWA, the set of winners \( W(b, v) \in W(b, v) \) is selected by a deterministic rule (e.g., a lexicographic order). Let \( \pi^{\text{det}}(M^*) \) be such allocation rule\(^1\). Then \( Q_n^{\text{det}}(b, v) \) is nonincreasing in \( v_n \), keeping \( b \) and \( v_n \) constant. This, along with the payment rule in step 3 of MWA, implies that a winner pays the price that is the minimum value he needs to report to still win, keeping his bundle and the bids of everyone else fixed.

\(^1\)The allocation rule \( \pi^{\text{det}} \) must be consistent in the following sense: let \( v_n \) and \( \tilde{v}_n \) be such that \( v_n < \tilde{v}_n \), but \( \overline{w}_n(b_n, v_n) = \overline{w}_n(b_n, \tilde{v}_n) \), then \( P[ n \in W(b, v_n, v_\ldots) ] \leq P[ n \in W(b, \tilde{v}_n, v_\ldots) ] \) for any \( b \) and \( v_\ldots \).
C. Solution of the OAP

We now give a sufficient condition under which a solution of MOAP is also a solution of OAP. To this end, define the hazard rate ordering on two random variables as follows:

Definition 1: A nonnegative random variable $Z_1$ is said to be smaller than a nonnegative random variable $Z_2$ under the hazard rate order, denoted by $Z_1 \leq_h Z_2$, if:

$$\Pr[Z_1 > z | Z_1 > z] \leq \Pr[Z_2 > z | Z_2 > z],$$

(19)

for all $z$, such that $z \geq z_0$. Also, if $Z_1 \leq_h Z_2$, then $Z_1$ is also smaller than $Z_2$ under the first order stochastic dominance (FOSD). Hence, the hazard rate order is stricter than the FOSD.

It is natural to expect that if there are two bundles where one contains the other, then the larger bundle is likely to have a higher value. This is precisely captured by Assumption 1 below.

Assumption 1: Let $s, t \in B_n$ be such that $s \subseteq t$. Then the conditional random variable $(X_n | B_n = s)$ is smaller than the conditional random variable $(X_n | B_n = t)$ under the hazard rate order. Equivalently, for all $n \in N$, $s, t \in B_n$ such that $s \subseteq t$, and $1 \leq j \leq i \leq K_n$, $n$,

$$\sum_{i=j}^{K_n} p(x_n^{|s}|t) \leq \sum_{i=j}^{K_n} p(x_n^{|t}|t),$$

(20)

Proposition 4 and 5 below describe the main results of this paper.

Proposition 4: Let $s, t \in B_n$ be such that $s \subseteq t$. Then under Assumption 1, $w_n(s, x_n^{|s}) \geq w_n(t, x_n^{|s})$ for $1 \leq i \leq K_n$.

Proof: The proof appears in Appendix A.

Proposition 5: A deterministic MWA gives a solution of the OAP under Assumption 1.

Proof: Notice that OAP and MOAP differ only in their constraints, and the relaxed IC constraint (10) is a subset of the IC constraint (7). Hence, we only need to verify that the solution given by the deterministic MWA satisfies the IC constraint. We show that, under the deterministic MWA, the truthful declaration of the types is a weakly dominant strategy for the buyers.

Let the bid vector be $(b, v)$. Based on the reported bundles $b$, the conflict graph $G_b$ is constructed. The weights of the nodes of $G_b$ are the MVVs for the bid vector $(b, v)$. Consider a buyer $n$. Let his true type be $(b_n^*, v_n^*)$. Since buyers are single minded, it can be assumed that $b_n^* \supseteq b_n^*$, otherwise the payoff from misreporting a bundle can be at most zero, which is less than or equal to the payoff from reporting the bundle truthfully. Also, if buyer $n$ does not get his bundle by bidding $(b_n, v_n)$ (and hence payoff equal to zero) then truthful bidding (payoff at least zero) cannot be worse. Hence, we only need to analyze the case where buyer $n$ wins by bidding $(b_n, v_n)$ such that $b_n \supseteq b_n^*$. Since buyer $n$ is a winner, there is a maximum weight independent set (henceforth MWIS) in $G_b$ that contains node $n$. Because of a deterministic tie-breaking rule, buyer $n$ pays the minimum value he needs to report to win. This is his value $x_n^+$ at which the value of the MWIS containing node $n$ exceeds the value of all MWIS not containing node $n$. Now, if instead buyer $n$ reports $b_n^*$, it can result in deletion of some edges incident on node $n$ in $G_b$, but cannot add any new edge. At the same time, from Proposition 4, the weight of node $n$ (or the MVV of buyer $n$) can possibly increase but cannot decrease. Hence, the value of the MWIS containing node $n$ can possibly go up but cannot decrease, while the value of the MWIS not containing node $n$ does not change. Buyer $n$ still wins and the payment if he declares $(b_n^*, v_n)$ cannot be more than what he pays when he declares $(b_n, v_n)$. Thus, truthful reporting of the bundle is a weakly dominant strategy.

We can now assume that buyer $n$ reports his bundle $b_n^*$ truthfully. Since the price that he pays only depends on his reported bundle and the bids of everyone else, but not on his reported value, truthful reporting of the value is a weakly dominant strategy. This completes the proof.

IV. DISCUSSION

Reserve prices: Given a bid vector $(b, v)$, the MWA does not include any buyer $n$ with $\pi_n(b_n, v_n) < 0$ in the set of winners $W(b, v)$. Depending on the tie-breaking rule, a buyer $n$ with $\pi_n(b_n, v_n) = 0$ may or may not be included in the set of winners. Assume that only buyers with $\pi_n(b_n, v_n) > 0$ are considered. Since $\pi_n(b_n, x_n^{|i}) \leq \pi_n(b_n, x_n^{|i+1})$, the seller equivalently sets (discriminatory) reserve prices for each buyer $n$. Given $b_n$, if the reserve price for buyer $n$ is equal to $x_n^{|i}$, then $w_n(b_n, x_n^{|i}) \leq 0$ for $1 \leq i \leq k - 1$. Hence, $h_n = \min_{0 \leq i \leq K_n} h_n^{|i} = \min_{0 \leq i \leq K_n} h_n^{|b_n^*|i}$. The reserve price for buyer $n$, denoted by $r_n(b_n)$, is given by:

$$r_n(b_n) = \max \left\{ v_n : v_n \in \arg\max_{v_n \in X_n} \hat{v}_n \Pr[X_n \geq \hat{v}_n | B_n = b_n] \right\},$$

(21)

Graphically, this corresponds to the y-intercept of the line through the lowermost point of the graph and the point $(1, 0)$ ($x^3$ in Figure 1).

On the hazard rate order assumption: In the absence of Assumption 1, the solution given by the MWA (under any tie-breaking rule) need not satisfy the IC constraint. Consider two buyers $\{1, 2\}$ and two items $\{A, B\}$. Buyer 1 is interested in bundle $\{A\}$ and has value $\$1$ for it. Buyer 2 can be interested in bundle $\{A\}$ or bundle $\{A, B\}$, each with probability $1/2$. For bundle $\{A\}$, his values can be $\$2$ or $\$4$, each with probability $1/2$. For bundle $\{A, B\}$, his values can be $\$2$ or $\$4$, with probabilities $0.9$ and $0.1$ respectively. It can be verified that if buyer 2 reports his type as $\{(A), \$4\}$ then he gets bundle $\{A\}$ at the price $\$4$, while if he reports his type as $\{(A, B), \$4\}$, he gets bundle $\{A, B\}$ at the price $\$2$. Thus, if the true type of buyer 2 is $\{(A), \$4\}$, he will misreport it to $\{(A, B), \$4\}$.

Implementation complexity: The deterministic MWA requires finding a maximum weight independent set in the conflict graph which is NP-hard. However, similar to $[1]$. 


a greedy scheme that allocates the bundles to the buyers according to the order induced by \( \overline{w}_n(b_i, v_n)/\sqrt{|b_n|} \) can be shown to achieve \( \sqrt{5} \) approximation\(^2\) of the revenue generated by the deterministic MWA.

V. CONCLUSIONS

We characterized a Bayesian revenue optimal multiple items auction with single-minded buyers under an intuitive partial hazard rate order assumption on the conditional distribution of any buyer's valuation. The resulting auction has a simple structure - the set of winners are the maximum weight independent set of the conflict graph of the buyers, and the payment made by a winner is the minimum value he needs to report to win. The contributions here provide a step towards understanding optimal auction problems where buyers' private information is multidimensional.

APPENDIX

PROOF OF PROPOSITION 4

Define \( F_{X_n|b_n}(z) \triangleq \mathbb{P}[X_n < z | B_n = b_n] \). We start with the following lemmas:

**Lemma 2:** For all \( n \in \mathcal{N} \) and \( b_n, \overline{w}_n(b_n, x_n^i) < x_n^i \) for \( 1 \leq i \leq K_n - 1 \), and \( \overline{w}_n(b_n, x_n^{K_n}) = x_n^{K_n} \).

**Proof:** This follows from the graphical construction of \( \overline{w}_n \)'s, as described in Section III-B. Details are omitted because of space constraints.

**Lemma 3:** For all \( n \in \mathcal{N}, b_n \), and \( 1 \leq i \leq K_n - 1 \),

\[
\overline{w}_n(b_n, x_n^i) = \arg \min_{c < x_n^i} \max_{z \in [x_n^i, x_n^{K_n}]} (z - c)(1 - F_{X_n|b_n}(z)),
\]

where the convention that if more than one value of \( c \) minimizes the maximum, then the largest such \( c \) is selected.

**Proof:** From (15), for any \( 1 \leq i \leq K_n \),

\[
(g_n^{b_n,i}, h_n^{b_n,i}) = (F_{X_n|b_n}(x_n^i), -x_n^i(1 - F_{X_n|b_n}(x_n^i))).
\]

Let \( I(z) \triangleq (F_{X_n|b_n}(z), -z(1 - F_{X_n|b_n}(z))) \) for \( z \in [x_n^i, x_n^{K_n}] \). Consider the plot of points \( I(z) \)'s. The convex hull of the points \( I(z) \) is the same as that of the points \( (g_n^{b_n,i}, h_n^{b_n,i}) \) for \( 0 \leq i \leq K_n - 1 \). Thus, \( \overline{w}_n \)'s are obtained as the slopes of the convex hull of points \( I(z) \)'s. Fix \( x_n^i \) for some \( 1 \leq i \leq K_n - 1 \). Call the line from \((0, -x_n^i)\) to \((1, 0)\) the line for \( x_n^i \). Given \( z \) and \( c < x_n^i \), consider the line through the point \( I(z) \) with slope \( c \), and let \( J \) be the point of intersection of this line with the line for \( x_n^i \). Then, \( (z - c)(1 - F_{X_n|b_n}(z))/x_n^i - c \) is the horizontal distance of \( J \) from the vertical line at \( (1, 0) \). Taking the maximum over \( z \) corresponds to the point \( J \) which is the intersection of the line of slope \( c \) that is tangent to the plot, and the line for \( x_n^i \). Then the minimizing \( c \) is the slope of the tangent at the point \( J \)' where the convex hull of \( I(z) \) intersects the line for \( x_n^i \), and hence same as \( \overline{w}_n(b_n, x_n^i) \). If there is more than one intersection point, the largest is selected. From Lemma 2, the minimum is achieved by \( c < x_n^i \). \( \blacksquare \)

For \( c < x_n^{K_n} \), define:

\[
\Phi_{X_n|b_n}(c) \triangleq \max_{z \in [x_n^i, x_n^{K_n}]} (z - c)(1 - F_{X_n|b_n}(z)).
\]

Notice that \( \Phi_{X_n|b_n}(c) \) is nonincreasing in \( c \).

**Lemma 4:** Let \( s, t \in B_n \) be such that \( s \leq t \). Then under Assumption 1, \( \Phi_{X_n|s}(c)/\Phi_{X_n|t}(c) \) is nonincreasing in \( c \).

**Proof:** Fix \( c_1 < c_2 < x_n^{K_n} \). We need to prove that:

\[
\frac{\Phi_{X_n|s}(c_1)}{\Phi_{X_n|t}(c_1)} \leq \frac{\Phi_{X_n|s}(c_2)}{\Phi_{X_n|t}(c_2)}.
\]

Under Assumption 1, \((1 - F_{X_n|s}(z))/(1 - F_{X_n|t}(z))\) is nonincreasing in \( z \). Let \( z_1^s \) and \( z_2^s \) denote the values of \( z \) achieving the maximum in the definition of \( \Phi_{X_n|s}(c_1) \) and \( \Phi_{X_n|s}(c_2) \) respectively. Clearly, \( z_1^s \geq c_1 \) and \( z_2^s \geq c_2 \). If \( z_1^s \leq z_2^s \),

\[
\Phi_{X_n|s}(c_1) = \frac{(z_1^s - c_1)(1 - F_{X_n|s}(z_1^s))}{1 - F_{X_n|t}(z_1^s)} \geq \frac{(z_1^s - c_2)(1 - F_{X_n|s}(z_1^s))}{1 - F_{X_n|t}(z_1^s)} \geq \frac{(z_2^s - c_2)(1 - F_{X_n|s}(z_2^s))}{1 - F_{X_n|t}(z_1^s)} \geq \Phi_{X_n|s}(c_2).
\]

On the other hand, if \( z_1^s \geq z_2^s \),

\[
\Phi_{X_n|s}(c_1) \geq \frac{(z_2^s - c_1)(1 - F_{X_n|s}(z_2^s))}{1 - F_{X_n|t}(z_2^s)} \geq \Phi_{X_n|s}(c_2).
\]

In either case, the required inequality is proved. Combining Lemma 3 and Lemma 4, for any \( c \leq \overline{w}_n(t, x_n^i) \), and \( 1 \leq i \leq K_n - 1 \),

\[
\frac{\Phi_{X_n|s}(c)}{x_n^i} \geq \frac{\Phi_{X_n|s}(c)\Phi_{X_n|s}(\overline{w}_n(t, x_n^i))}{(x_n^i - c)(\Phi_{X_n|s}(\overline{w}_n(t, x_n^i)))} \geq \frac{\Phi_{X_n|s}(\overline{w}(t, x_n^i))\Phi_{X_n|s}(\overline{w}(t, x_n^i))}{x_n^i - \overline{w}(t, x_n^i)} = \Phi_{X_n|s}(\overline{w}(t, x_n^i)),
\]

where the first inequality is by Lemma 4, and the second is because \( \Phi_{X_n|b_n}(c) \) is nonincreasing in \( c \). Hence, from Lemma 3, it follows that \( \overline{w}_n(s, x_n^i) \geq \overline{w}_n(t, x_n^i) \) for \( 1 \leq i \leq K_n - 1 \). Also \( \overline{w}_n(s, x_n^{K_n}) = \overline{w}_n(t, x_n^{K_n}) = x_n^{K_n} \). This completes the proof of Proposition 4.

REFERENCES


